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The Maximum Principle for Discrete Control Systems with Time Lag

In the paper one formulates the Maximum Principle for discrete control systems which contain time lags for the state variables, as well as for the control variables. The proof of the Maximum Principle is based on Hestenes' multiplier-theorem.

The mathematical models of many economic problems reduce to control systems with discrete variables in which time lags appear in the state variables and / or in the control variables.

In the following we present the Maximum Principle for control systems with discrete variables in which time lags appear in the state variables as well as in the control variables.

The proof of the Principle is similar to that given in [3] for control systems with continuous variables and it is based on Hestenes'multiplier-theorem.

1. THE MODEL

We consider a control system of the form:

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-s}, u_n, u_{n-1}, \ldots, u_{n-s}, b) \tag{1}$$

with

$$x_{n_0-i} = \alpha_{n_0-i}(b), \quad i = 0, 1, \ldots, s \tag{2}$$

$$\mu_{n_0-i} = \eta_{n_0-i} \qquad i = 0, 1, \ldots, s \tag{3}$$

$$x_N = \omega(b) \tag{4}$$

and the functionals

$$I_\gamma(u, b) = g_\gamma(b) + \sum_{k=n_0}^{N-1} L_{\gamma k}(x_k, x_{k-1}, ..., x_{k-s}\, u_k, u_{k-1}, ..., u_{k-s}, b) \qquad (5)$$

$$\gamma = 0, 1, ..., p$$

We noted by x_n the vector of state variables at moment n, by u_n the vector of control variables at moment n and b is the vector of control parameters. We have:

$$x_n \in \mathbf{R}^q; \quad b \in B \subset \mathbf{R}^l; \quad u_n \in U \subset \mathbf{R}^h$$

$$n = 0, 1, ..., N-1$$

Functions:

$$f_n : \mathbf{R}^q \times U \times B \to \mathbf{R}^q$$

$$L_{\gamma k} : \mathbf{R}^q \times U \times B \to \mathbf{R}$$

$$g_\gamma : B \to \mathbf{R}; \quad \alpha_{n_0-i} : B \to \mathbf{R}^q; \quad i = 0, 1, ..., s$$

$$\omega : B \to \mathbf{R}^q$$

are given.

Definition 1

The sequence $\{u_n, b\}_{n_0-s}^{N-1}$ is an admissible control if:

a) $u_{n_0-i} = \eta_{n_0-i} \quad i = 1, 2, ..., s$

b) $u_n \in U; \quad n = 0, 1, ..., N-1$

c) Let $\{x_n\}_{n_0-s}^{N}$ be the corresponding solution of the dynamic equation with

$$x_{n_0-i} = \alpha_{n_0-i}(b); \quad i = 0, 1, ..., s$$

The triplet

$$\{u_n, b; x_n\}_{n_0-s}^{N-1}$$

satisfies conditions:

$$I_\gamma \leqslant 0 \quad \gamma = 1, 2, ..., p'$$

$$I_\gamma = 0 \quad \gamma = p' + 1, ..., p$$

and, moreover, the following final condition is satisfied:

$$x_N = \omega(b)$$

Definition 2

The control $\{\tilde{u}_n, \tilde{b}\}_{n_0-s}^{N-1}$ is optimal control if it minimizes functional I_0 in the class of all admissible controls.

2. THE MAXIMUM PRINCIPLE

Let $\{\tilde{u}_n, \tilde{b}\}_{n_0-s}^{N-1}$ be an optimal control and $\{\tilde{x}_n\}_{n_0-s}^{N}$ the corresponding solution.

We introduce the following notations:

$$A_n = \frac{\partial f_n}{\partial x}(\tilde{x}_n, \tilde{x}_{n-1}, \ldots, \tilde{x}_{n-s}, \tilde{u}_n, \tilde{u}_{n-1}, \ldots, \tilde{u}_{n-s}, b) \tag{6}$$

$$B_{n+1}^j = \frac{\partial f_n}{\partial y_j}(\tilde{x}_n, \tilde{x}_{n-1}, \ldots, \tilde{x}_{n-s}, \tilde{u}_n, \tilde{u}_{n-1}, \ldots, \tilde{u}_{n-s}, b) \tag{7}$$

$$C_{\gamma,n} = \frac{\partial L_{\gamma,n}}{\partial x}(\tilde{x}_n, \tilde{x}_{n-1}, \ldots, \tilde{x}_{n-s}, \tilde{u}_n, \tilde{u}_{n-1}, \ldots, \tilde{u}_{n-s}, \tilde{b}) \tag{8}$$

$$D_{\gamma,n+1}^j = \frac{\partial L_{\gamma,n}}{\partial y_j}(\tilde{x}_n, \tilde{x}_{n-1}, \ldots, \tilde{x}_{n-s}, \tilde{u}_n, \tilde{u}_{n-1}, \ldots, \tilde{u}_{n-s}, \tilde{b}) \tag{9}$$

where f_n and $L_{\gamma,n}$ are written as functions of

$$(x, y_1, y_2, \ldots, y_s, u, v_1, v_2, \ldots, v_s, b)$$

Theorem 1 (the Maximum Principle)

Let $\{\tilde{u}_n, \tilde{b}\}_{n_0-s}^{N-1}$ be an optimal control and $\{\tilde{x}_n\}_{n_0-s}^{N}$ the corresponding solution.

a) There exist multipliers:

$$\lambda_0, \lambda_1, \ldots, \lambda_{p+q}$$

with

(i) $\sum_{i=0}^{p+q} |\lambda_j| \neq 0$

(ii) $\lambda_\gamma \geqslant 0 \quad \gamma = 0, 1, \ldots, p'$

(iii) $\lambda_\gamma I_\gamma(\tilde{u}, \tilde{b}) = 0 \quad \gamma = 1, 2, \ldots, p'$

b) Let $\tilde{\psi}_n$ be the solution of the adjoint system

$$\psi_{n-1} = \psi_n A_n + \sum_{j=0}^{s} \psi_{n+j+1} B_{n+j+1}^{j+1} - \sum_{\beta=0}^{p} \lambda_\beta \left[C_{\beta,n} + \sum_{j=0}^{s} D_{\beta,n+j+1}^{j+1} \right] \tag{10}$$

with: $\tilde{\psi}_{N-1}^i = -\lambda_{p+i}$, $\bar{\psi}_m \equiv 0$ for $n \geqslant N$

and $\mathcal{H}_n$

$$\mathcal{H}_n(x_{n+s}, x_{n+s-1}, \ldots, x_{n-s}, u_{n+s}, u_{n+s-1}, \ldots, u_{n-s}, b) =$$

$$= \sum_{j=0}^{s} H_{n+j}(x_{n+j}, x_{n+j-1}, \ldots, x_{n+j-s}, u_{n+j}, \ldots, u_{n+j-s}, b) \tag{11}$$

the Hamiltonian of the system, where

$$H_n(x_n, y_1, ..., y_s, u_n, v_1, ..., v_s, b) = \tilde{\psi}_n f_n(x_n, y_1, ..., y_s, u_n, v_1, ..., v_s, b) -$$

$$- \sum_{\beta=0}^{p} \lambda_\beta L_{\beta n}(x_n, y_1, ..., y_s, u_n, v_1, ..., v_s, b) \tag{12}$$

We have the inequality (The Maximum Principle)

$$\frac{\partial \mathcal{H}_n}{\partial u}(\tilde{x}_{n+s}, ..., \tilde{x}_{n-s}, \tilde{u}_{n+s}, ..., \tilde{u}_{n-s}, \tilde{b})[u - \tilde{u}_n] \leqslant 0 \tag{13}$$

for any $u \in U$ and $n = n_0, n_0 + 1, ..., N - 1$.
(c) The transversality conditions are:

$$\sum_{\rho=0}^{p} \lambda_\rho \frac{\partial g_\rho(\tilde{b})}{\partial b^\sigma} + \tilde{\psi}_{N-1} \frac{\partial \omega(\tilde{b})}{\partial b^\sigma} - \tilde{\psi}_{n_0-1} \frac{\partial \alpha_{n_0}(\tilde{b})}{\partial b^\sigma} -$$

$$- \sum_{k=n_0}^{N-1} \frac{\partial H_k}{\partial b^\sigma}(\tilde{x}_k, ..., \tilde{x}_{k-s}, \tilde{u}_k, ..., \tilde{u}_{k-s}, \tilde{b}) = 0 \tag{14}$$

3. The proof of the Maximum Principle

In order to prove the Maximum Principle we transform the given problem into an equivalent one, for which one applies easily Hestenes' multiplier-theorem.

We state first Hestenes' multiplier-theorem.

Difinition 3

Let be functionals:

$$J_\rho : \mathcal{C} \to \mathcal{R};\ \rho = 0,1,..., m$$

and $$\tilde{Z} \in \mathcal{C}$$

A set K of $m + 1$ — dimensional vectors is called a *derivative set* for $\{J_\rho\}$ in point $\tilde{Z}$ if, for any set of vectors $k_1, k_2, ..., k_h \in K$ there exists a function

$$Z : [0, \delta]^h \to \mathcal{C}$$

such that $Z(0) = \tilde{Z}$ and:

$$J_\rho(Z(\varepsilon)) - J_\rho(\tilde{Z}) = \sum_{j=1}^{h} \varepsilon_j k_j^\rho + 0(\varepsilon) \tag{15}$$

for $\varepsilon \in [0, \delta]^h$.

Theorem 2. (Hestenes' multiplier-theorem)

Let K be a derivative set for the system of functionals $\{J_\rho\}_0^m$ in point $\tilde{Z} \in \mathcal{C}$. If $\tilde{Z}$ realizes the minimum of functional J_0 in the class of elements $Z \in \mathcal{C}$ which satisfy conditions:

$$J_\rho(Z) \leqslant 0\,;\ \rho = 1, 2, \ldots, m'$$

$$J_\rho(Z) = 0\,;\ \rho = m' + 1, \ldots, m$$

then there exist multipliers:

$$\lambda_0,\ \lambda_1, \ldots,\ \lambda_m$$

such that

a) $\sum_{\rho=0}^{m} |\lambda_\rho| \neq 0$

b) $\lambda_\rho \leqslant 0\ \ \rho = 0, 1, \ldots, m'$

c) $\lambda_\rho J_\rho(\tilde{Z}) = 0$

for any $\rho = 1, 2, \ldots, m'$

and, moreover,

$$\sum_{\rho=0}^{h} \lambda_\rho\, k^\rho \geqslant 0$$

for any $k \in K$.

Notations

In order to transform the control problem, we introduce the following new variables:

a) $q_{\gamma,n}$ defined by the system

$$q_{\gamma,n-1} = q_{\gamma,n}\, A_{n-1} + \sum_{j=1}^{s} q_{\gamma,n+j}\, B^j_{n+j} + C_{\gamma,n-1} + \sum_{j=1}^{s} D^j_{\gamma,n+j} \qquad (16)$$

with conditions $q_{\gamma,n} \equiv 0$ for $n \geqslant N$

b) $P_{i,n}$ defined by the system

$$P_{i,n-1} = P_{i,n} \cdot A_{n-1} + \sum_{j=1}^{s} P_{i,n+j} \cdot B^j_{n+j} \qquad (17)$$

with conditions: $P^l_{i,N} = \delta^l_i = \begin{cases} 0 & i \neq l \\ 1 & i = l \end{cases}$

$$P_{i,n} \equiv 0$$

for $n > N$

c) Functions $F_{\gamma,n}$ and $G_\gamma(b)$, defined for $\gamma = 0, 1, \ldots, p$ as follows:

$$F_{\gamma,n}(x, y_1, \ldots, y_s, u, v_1, \ldots, v_s, b) = L_{\gamma,n}(x, y_1, \ldots, y_s, u, v_1, \ldots, v_s, b) -$$

$$- C_{\gamma,n} \cdot x - \sum_{j=1}^{s} D^j_{\gamma,n+1} y_j + q_{\gamma,n+1}[f_n(x, y_1, \ldots, y_s, u, v_1, \ldots, v_s, b) - A_n \cdot x -$$

$$- \sum_{j=1}^{s} B^j_{n+1} y_j] \tag{18}$$

$$G_\gamma(b) = g_\gamma(b) + q_{\gamma,n_0} \cdot \alpha_{n_0}(b) + \sum_{j=1}^{s} \sum_{k=n_0}^{n_0+s-1} [q_{\gamma,k+1} \cdot B^j_{k+1} + D^j_{\gamma,k+1}] \cdot x_{k-j} \tag{19}$$

and for $i = 1, 2, \ldots, q$ they are defined as follows:

$$F_{p+i.n}(x, y_1, \ldots, y_s, u, v_1, \ldots, v_s, b) = P_{i,n+1}[f_n(x, y_1, \ldots, y_s, u, v_1, \ldots, v_s, b) -$$

$$- A_n x - \sum_{j=1}^{s} B^j_{n+1} \cdot y_j]. \tag{20}$$

Remark 1

A direct computation shows that the following relations are satisfied:

$$\frac{\partial F_{\gamma,n}}{\partial x}(\tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_s, \tilde{u}_n, \tilde{v}_1, \ldots, \tilde{v}_s, \tilde{b}) = 0$$

$$\frac{\partial F_{\gamma.n}}{\partial y_j}(\tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_s, \tilde{u}_n, \tilde{v}_1, \ldots, \tilde{v}_s, \tilde{b}) = 0$$

for $\gamma = 0, 1, \ldots, p + q$ and $j = 1, 2, \ldots, s$.

Proof of theorem 1

In order to prove theorem 1 we introduce the following functionals:

$$J_\gamma = G_\gamma(b) + \sum_{k=n_0}^{N-1} F_{\gamma,k}(x_k, \ldots, x_{k-s}, u_k, \ldots, u_{k-s}, b); \tag{21}$$

$$\gamma = 0, 1, \ldots, p, p+1, \ldots, p+q$$

By a direct computation, laborious enough *), it follows

$$\begin{cases} J_\gamma = I_\gamma; & \gamma = 0, 1, \ldots, p \\ J_{p+i} = x^i_N - \omega^i(b); & i = 1, 2, \ldots, q \end{cases}$$

Thus, the initial problem, in which final conditions $x_N = \omega(b)$ appear, was transformed in a problem without final conditions, these conditions having been replaced by conditions of the form $J_{p+i} = 0$.

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*) The whole computation can be found in [1]

The initial control problem is equivalent to that of minimizing functional J_0 in the class of controls for which we have:

$$J_\gamma \leqslant 0 \quad \gamma = 1,2, \ldots, p'$$

$$J_\gamma = 0 \quad \gamma = p' + 1, \ldots, p + q.$$

We shall apply to the later Hestenes' multiplier theorem.

In order to do so, we consider the set K (see definition 3) consisting of vectors of the form

$$k^\rho = \sum_{j=1}^{s} \frac{\partial F_{\rho,n+j}}{\partial u_{n+j}} (\tilde{x}_{n+j}, \ldots, \tilde{x}_{n+j-s}\, \tilde{u}_{n+j}, \ldots, \tilde{u}_{n+j-s}, \tilde{b}) \cdot [u - u_n] \tag{23}$$

$$u \in U, \; n = n_0, \ldots, N-1$$

to which we add the vectors of the form:

where
$$k^\rho = \sum_\alpha \bar{k}^\rho_\alpha \cdot m^\alpha \tag{24}$$

$$\bar{k}^\rho_\alpha = \frac{\partial G_\rho(\tilde{b})}{\partial b^\alpha} + \sum_{k=n_0}^{N-1} \frac{\partial F_{\rho,k}}{\partial b^\alpha} (\tilde{x}_k, \ldots, \tilde{x}_{k-s}, \tilde{u}_k, \ldots, \tilde{u}_{k-s}, \tilde{b}).$$

Let us verify that we have indeed a derivative set. Let us choose

$$k_0, \; k_1, \ldots, \; k_r \in K$$

$$k_0^\rho = \sum \bar{k}^\rho_\alpha \, m^\alpha$$

$$k_i^\rho = \sum_{j=0}^{s} \frac{\partial F_{\rho,n_i+j}}{\partial u} (\tilde{x}_{n_i+j}, \ldots, \tilde{x}_{n_i+j-s}, \tilde{u}_{n_i+j}, \ldots, \tilde{u}_{n_i+j-s}\, \tilde{b}) \cdot [u_i - \tilde{u}_{n_i}]; \; i = 1, 2, \ldots, r$$

We choose the couple $(u_n(\varepsilon), b(\varepsilon))$ such that

$$u_n(\varepsilon) = \tilde{u}_n \quad n \neq n_j$$

$$u_{n+j}(\varepsilon) = \tilde{u}_{n_i+j} + \sum_{n_l = n_i + j} [u_{l+j} - \tilde{u}_{n_i+j}) \, \varepsilon_l$$

for $n = n_i, \; j = 0,1,\ldots, \; s$ and $0 \leqslant \varepsilon_l \leqslant \dfrac{1}{r}$, $b(\varepsilon) = \tilde{b} + \varepsilon_0 \, m$

where:

$$\varepsilon = (\varepsilon_0, \ldots, \varepsilon_r).$$

We note that, assuming the set U, convex $u_n(\varepsilon) \in U$; for $n = n_0, 1, \ldots, N-1$

Let:

$$\varphi_\rho(\varepsilon) = J_\rho(u_n(\varepsilon), \; b(\varepsilon)) - J_\rho(\tilde{u}_n, \; \tilde{b})$$

Taking into account that, according to remark 1,

$\dfrac{\partial F_{\rho,k}}{\partial x_k}$ and $\dfrac{\partial F_{\rho,k}}{\partial y_j}$ vanish along the optimal solution, it follows from a direct

computation*) that:

$$\frac{\partial \varphi_\rho(0)}{\partial \varepsilon_0} = k_0^\rho$$

$$\frac{\partial \varphi_\rho(0)}{\partial \varepsilon_i} = k_i^\rho$$

Hence it follows that K is a derivative set for functionals $\{J_\rho\}$.

Therefore, according to Hestenes' multiplier-theorem, there exist multipliers λ_ρ such that

$$\sum \lambda_\rho k^\rho \geqslant 0 \quad \text{for any } k \in K.$$

Hence, it follows that

$$\sum_{\rho=0}^{n+q} \lambda_\rho \sum_{j=0}^{s} \frac{\partial F_{\rho,n+j}}{\partial u} (\tilde{x}_{n+j}, \ldots, \tilde{x}_{n+j-s}, \tilde{u}_{n+j}, \ldots, \tilde{u}_{n+j-s}, \tilde{b})[u - \tilde{u}_n] \geqslant 0 \quad (25)$$

for any $u \in U$ and $n = n_0, n_{0+1}, \ldots, N-1$.

We have also:

$$\sum_{\rho=0}^{p+q} \lambda_\rho \sum_{\alpha} \left[\frac{\partial G_\rho(\tilde{b})}{\partial b^\alpha} + \sum_{k=n_0}^{N-1} \frac{\partial F_{\rho,k}}{\partial b^\alpha} (\tilde{x}_k, \ldots, \tilde{x}_{k-s}, \tilde{u}_k, \ldots, \tilde{u}_{k-s}, \tilde{b}) \right] \cdot m^\alpha \geqslant 0.$$

Since the above relation is satisfied for any vector m, it follows

$$\sum_{\rho=0}^{p+q} \lambda_\rho \left[\frac{\partial G_\rho(\tilde{b})}{\partial b^\alpha} + \sum_{k=n_0}^{N-1} \frac{\partial F_{\rho,k}}{\partial b^\alpha} (\tilde{x}_k, \ldots, \tilde{x}_{k-s}, \tilde{u}_k, \ldots, \tilde{u}_{k-s}, \tilde{b}) \right] = 0. \quad (26)$$

In order to express this result in terms of the initial problem, we note

$$\tilde{\psi}_n = -\sum_{i=1}^{q} \lambda_{p+i} \cdot P_{i,n+i} - \sum_{\beta=0}^{p} \lambda_\beta \cdot q_{\beta,n+1}.$$

By direct computation, it follows:

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n \cdot A_n + \sum_{j=1}^{s} \tilde{\psi}_{n+j} \cdot B_{n+j+1}^{j+1} - \sum_{\beta=0}^{p} \lambda_\beta [C_{\beta,n} + \sum_{j=0}^{s} D_{\beta,n+j+1}^{j+1}].$$

Since $q_{\beta,N} = 0$ and $P_{i,N}^j = \delta_i^j$ it follows

$$\tilde{\psi}_{N-1}^i = -\lambda_{p+i}.$$

A direct computation shows that the Hamiltonian in the second part (formula 11) can be written in the following way:

$$\mathcal{H}_n(\tilde{x}_{n+s}, \ldots, \tilde{x}_{n-s}, \tilde{u}_{n+s}, \ldots, \tilde{u}_{n+1}, u, \tilde{u}_{n-1}, \ldots, \tilde{u}_{n-s}, \tilde{b}) =$$

$$= -\sum_{j=0}^{s} \sum_{\rho=0}^{p+q} \lambda_\rho F_{\rho,n+j}(\tilde{x}_{n+j}, \ldots, \tilde{x}_{n+j-s}, \tilde{u}_{n+j}, \ldots, u, \ldots, \tilde{u}_{n+j-s}, \tilde{b}) +$$

$$+ \sum_{j=0}^{s} \tilde{\psi}_{n+j} \left[A_{n+j} \cdot \tilde{x}_{n+j} + \sum_{l=0}^{s} B_{n+j+1}^{l} \cdot \tilde{x}_{n+j-l} \right] -$$

*) For the details see [1]

$$-\sum_{j=0}^{s}\sum_{\beta=0}^{p}\lambda_\rho(C_{\beta,n+j}\cdot\tilde{x}_{n+j}+\sum_{l=1}^{s}D_{\beta,n+l+j}\cdot\tilde{x}_{n+j-l})$$

whence it follows:

$$\frac{\partial\mathcal{H}_n}{\partial u}\tilde{x}_{n+s},\ldots,\tilde{x}_{n-s},\tilde{u}_{n+s},\ldots,\tilde{u},\ldots,\tilde{u}_{n-s},\tilde{b})=$$

$$=\sum_{j=0}^{s}\sum_{\rho=0}^{p+q}\lambda_\rho\frac{\partial F_{\rho,n+j}}{\partial u}(\tilde{x}_{n+s},\ldots,\tilde{x}_{n+j-s},\tilde{u}_{n+j},\ldots,\tilde{u}_{n+j-s},\tilde{b})$$

Taking into account relation (25), it follows that:

$$\frac{\partial\mathcal{H}_n}{\partial u}(\tilde{x}_{n+s},\ldots,\tilde{x}_{n-s},\tilde{u}_{n+s},\ldots,\tilde{u}_{n-s},\tilde{b})[u-\tilde{u}_n]\leqslant 0$$

that is the Maximum Principle.

From relation (26), expressing functions G_ρ and $F_{\rho,k}$ in terms of the initial problem, we obtain the transversality condition of Theorem 1.

Remark

For $s=0$, Theorem 1 gives the Maximum Principle for control systems without time lags.

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LE PRINCIPE DU MAXIMUM POUR LES SYSTEMES DE COMMANDE A RETARDS

L'auteur présente le Principe du Maximum pour les systèmes discrets de commande contenant des retards, tant pour les variables d'état, que pour celle de commande.

La démonstration du Principe du Maximum a pour base le théorème desmultiplicateurs de Hestenes

ПРИНЦИП МАКСИМУМА ДИСКРЕТНЫХ УПРАВЛЕНЧЕСКИХ СИСТЕМ С ЗАДЕРЖКАМИ

В этом материале представляется принцип максимума дискретных управленческих систем, которые отличаются задержками как для переменных состояния так и для переменных управления. Доказание принципа основывается на теореме множителей Хестенеса.