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Some Algorithms for Nonlinear Integer Programming

The present study makes some modifications in Kelley's cutting plane method [1] *for nonlinear integer programming and in the lexicographical method of Korte, Krelle and Oberhofer* [2] *for linear integer programming. The two modified algorithms are combined in several modes and the variants obtained are tested on some classical examples from literature* [5].

In conclusion of his work [1], Kelley mentions the possibility for the use of his method in solving nonlinear programming problems with integer variables.

For this, Kelley suggests that Gomory's method [3] be used at each iteration. All through the years this method has proved less efficient than the methods based on the branch-and-bound principle. Thus, most computer codes are based on branch-and-bound type methods.

Besides the Kelley type cutting planes, called here "external" (constructed by means of a point exterior to the feasible domain and which does not intersect this domain), cutting planes called "internal" (built by means of feasible points, planes which intersect the domain of feasible points) are also constructed in this study.

In contradistinction to Kelley who proposes the solving of the auxiliary problems at each iteration optimal the study outlines the advantages of solving these problems only suboptimally.

In the second part of the study modifications are made in the lexicographical method of Korte, Krelle and Oberhofer [2], by using the special structure of the auxiliary problems in Kelley's method. The possibility for the re-work of the procedure of lexicographical enumeration of this method will be used.

The two algorithms, thus modified, can be used in obtaining new algorithms of nonlinear integer programming.

The results of the testing of the variants of the cutting plane method on several problems known in literature (Lawler and Bell [5]) are given in conclusion of the study.

KELLEY'S CUTTING PLANE METHOD AND ITS MODIFICATIONS

The problem treated by Kelley [1] is the following:

$$min\ f(x)$$

$$g_i(x) \leqslant 0, \quad i = 1, ..., m,\ x \text{ integer} \tag{1}$$

where: $f(x)$ and $g_i(x)$, $i = 1, ..., m$ are convex functions and $x = (x_1, ..., x_n)$.

Problem (1) changes into an equivalent one

$$min\ x_{n+1}$$

$$f(x) - x_{n+1} \leqslant 0$$

$$g_i(x) \leqslant 0, \quad i = 1, ..., m,\ x \text{ integre} \tag{2}$$

The feasible domain is limited and contained in a polyhedron in which functions $f(x)$ and $g_i(x)$, $i = 1,..., m$ are differentiable with bounded partial derivates.

Let

$$P^\circ = \{x \mid Ax \leqslant b, \quad x \geqslant 0\}$$

be the polyhedron containing the feasible domain of problem (2).

We denote $g_0(x) := f(x) - x_{n+1}$.

The method proposed by Kelley for solving problem (2) consists of the following steps:

1. $i := 0$ is considered
2. Gomory's method [3] is applied for problem

$$z^i := min\ x_{n+1}$$

$$x \in P^i \tag{3}$$

$$x \text{ integer.}$$

Consider (x^i, z^i) the solution of the above problem.

If $x^i \notin \{x \mid g_k(x) \leqslant 0, k = 0, m\} = G$ we construct a cutting plane of the form

$$x \nabla g_j(x^i) \leqslant x^i \nabla g_j(x^i) - g_j(x^i), \tag{4}$$

where:

$$g_j(x^i) = max\ g_k(x^i)$$

$$k = 0, m.$$

3. We construct

$$P^{i+1} = P^i \cap \{x \mid x \triangledown g_j(x^i) \leqslant x^i \triangledown g_j(x^i) - g_j(x^i)\} \tag{4}$$

and repeat from step 2.

The number of constraints of problems (3) is growing with the number of iterations made. The cutting planes (4) built by means of the points exterior to the feasible domain G are called "external" cutting planes.

The modifications in Kelley's method consist in the following.

1. Problem (3) is not solved with the aid of Gomory's method [3] but with another more efficiently algorithm of the branch-and-bound type. We shall use here the lexicographical searching method of Korte, Krelle and Oberhofer [2].

2. Problems (3) are not solved optimally at each iteration because any point exterior to the admissible field $x^i \notin G$ may generate a Kelley type "external" cutting plane.

3. Cutting planes are constructed also if $x^i \in G$. We call these cutting planes "internal" and they are of the form:

$$x_{n+1} \leqslant f(x^i).$$

As we have $f(x^{i+1}) \leqslant f(x^i)$, these cutting planes do not increase the dimension of the problems, they are parallel between them and thus those constructed up to iteration i become superfluous for the next iterations.

The algorithm stops when P^i becomes empty (no longer contains any point with integer coordinates). The last feasible solution identified will also be the optimal solution of problem (2).

KORTE'S LEXICOGRAPHICAL SEARCHING METHOD AND ITS MODIFICATIONS

The problem treated by Korte, Krelle and Oberhofer [2] is:

$$\min cx$$

$$Ax \leqslant b \quad 0 \leqslant x \leqslant d, x \text{ integer} \tag{5}$$

where: A is a matrix and c, b and d corresponding size vectors. An upper bound S° of the minimum of problem (5) is used for solving problem (5).

Search feasible points of system (6) in a lexicographical order [2] by setting out from point $x^\circ := (0, \ldots, 0)$

$$\begin{gathered} cx \leqslant S^\circ \\ Ax \leqslant b \\ 0 \leqslant x \leqslant d. \end{gathered} \tag{6}$$

Every point x^i satisfying constraints (6) generates an "internal" cutting plane of the form

$$cx \leqslant S^i := cx^i - 1$$

which replaces plane $cx \leqslant S^{i-1}$ and thus it does not increase the size of problem (6).

The modifications in the Korte, Krelle and Oberhofer lexicographical searching method consist of:

1. Problem (3) can also be written as follows

$$\begin{gathered} min\ z \\ A_1x - z \leqslant \alpha_1 \\ A_2x \leqslant \alpha_2\ x \text{ integer} \end{gathered} \tag{7}$$

or under the form of a series of systems knowing for the beginning an upper bound of the minimum of problem (3)

$$\begin{gathered} A_1x - \alpha_1 \leqslant S^i \\ A_2x - \alpha_2 \leqslant 0\ x,\ \text{integer} \end{gathered} \tag{8}$$

where: $\bar{S}^i = (S^i, ..., S^i)$.

The algorithm of solving problem (7) is

a) consider $x^\circ := (0, ..., 0)$ and $r := 0$;

b) determine, if possible, x^r is the solution of system (8) and go to c) else to d);

c) calculate $S^r := max\ a_j^1 x^r$(where a_j^1 is the j-th line of A_1) and put $r := r + 1$ after which go to b);

d) if $r = 0$ problem (7) has no solutions otherwise x^r is the sought optimal solution.

Observation: Problem (7) without constraints $A_2x \leqslant \alpha_2$ is under the form of Benders's master problem for which the algorithm of Korte and others has been modified in [4].

2. As problems (7) are modified from an iteration to another only because some free terms diminish inequality $s^{i+1} < s^i$, being satisfied, the problem is completed with new constraints and, under these conditions the infeasible solutions of problem (8) for iteration i remain infeasible for the problems at the next iterations $i + 1$, $i + 2$, thus it is not necessary to set out for each iteration with point $x^\circ := (0,..., 0)$ but with the first admissible solution of the problem at the previous iteration.

THE VARIANTS OF THE COMBINED METHOD

Three variants of the cutting plane method which were tested on the Lawler and Bell [5] problems are described in the following.

a) The first variant consists in using Kelley's method in which the Gomory method for integer linear programming problems is replaced by the lexicographical searching method of Korte and others. Problems (7) are considered to be independent from an iteration to another and are solved optimally at each iteration.

b) The second variant is variant a) in which at each iteration the searching starts with lexicographic first fiseable point founded at the previsous iteration.

c) The third variant is variant b) in which the problems are not solved optimally at each iteration and the solving of a problem at iteration starts with the lexicographical first feasible solution of the problem at iteration $i-1$.

As the domain of feasible solutions is supposed to be bounded and the number of discrete points is finite and at each iteration at least one discrete point is separated, the above variants converge into a finite number of steps.

In all variants, the initial point exterior to the feasible domain and the initial polyhedron are automatically determined and the initially nonlinear feasible domain is supposed to be bounded, a hypothesis whose verification is difficult.

The initial point exterior to the feasible domain is considered

$$x_j^e := d_j + 1,\ j = 1,\ldots,\ n$$

where d_j is the upper bound of variable x_j and the initial polyhedron P^0 is determined as follows:

$$P^0 = \{x \mid x \nabla g_i(x^e) \leqslant x^e \nabla g_i(x^e) - g_i(x^e),\ \ i = 1,\ \ldots,\ m$$

where $x^e := (x_1^e,\ldots,\ x_n^e)$.

NUMERICAL RESULTS

The above variants were programmed in the FORTRAN IV language and the programmes were run on an IBM 370/145 computer of the Economic Cybernetics Department. The results of the test problems are given in table. 1.

Table 1

TEST PROBLEMS	Variant 1 no. it.	Variant 1 time	Variant 2 no. it.	Variant 2 time	Variant 3 no. it.	Variant 3 time
1. Lawler-Bell 1	4	3.78	4	3.73	5	2.64
2. Lawler-Bell 2	3	23.3	3	23.33	1	0.79
3. Lawler-Bell 3	1	4.78	1	4.76	0	1.64
4. Lawler-Bell 4	2	7.34	2	7.33	1	1.38
5. Lawler-Bell 5	16	220.06	16	219.88	41	21.88

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UN ALGORITHME DE PROGRAMMATION ENTIERE NON-LINEAIRE

L'auteur apporte quelques modifications à la méthode du plan de section de Kelley [1] pour la programmation entière non-linéaire et à la méthode lexicographique de Korte, Krelle et Oberhofer [2] pour la programmation entière non linéaire. Ces deux algorithmes sont combinés de plusieurs manières et les variantes obtenues sont soumises à l'épreuve en utilisant quelques exemples classiques de la littérature [5].

АЛГОРИТМ НЕЛИНЕЙНОГО ПОЛНОГО ПРОГРАММИРОВАНИЯ

В настоящей работе вносятся некоторые изменения методу плана подразделения Келлей [1] для нелинейного полного программирования и лексикографическому методу Корте Крелле и Оберхоферу [2] для полного нелинейного программирования. Оба логаритма скомбинированны в нескольких формах и достигнутые варианты подвергаются тестам на нескольких классических примерах литературы по специальности [5].