Prof. dr. G. THEILER
Bucharest University
Prof. dr. L. TŐVISSI
Department of Economic Cybernetics
Academy of Economic Studies

Limit Laws for Extreme Values, Weibull Distribution and Some Statistical Problems of Control Global Analysis

This study has as a starting point certain findings concerning some opinions on the character of universality of the Weibull distribution and consequently on the right to be used in the solving of the problems of global analysis of reception control. The theoretical support of those sharing this point of view is the special place occupied by the Weibull distribution in the statistics of order, or better said, between the limit laws of the extreme values.

As in so many other cases, the risk exists that by starting from certain situations in which the use of a distribution or another is justified, they should pass to others without a preceding careful analysis and they should draw some wrong conclusions. For a correct understanding of the problems a careful analysis is necessary of the conditions in which the limit laws for extreme values are valid and the extent to which they are applicable in case of a relatively small number of data. Or, the limit laws for extreme values are either not presented at all in the monographs and treatises of mathematical statistics or are presented with too big constraints and with insufficient accuracy as in Kendall and Stuart [1] or too shortly and imprecisely as in Gumbels monograph on extreme values [2]. As a rule, when the question is about limit laws on extreme values reference is made to a famous paper by Gnedenko [3] hardly accessible and which is hard to follow by nonspecialists. As a matter of fact this paper also makes reference to another paper by Hincin which is still more difficult to find.

Therefore, the authors are trying to present in this study the limit laws for extreme values in a form accessible to those concerned with the applications of mathematical statistics in economy, yet a complete presentation.

The mechanism of the limit laws for extreme values should be well understood because in most cases, a relatively small number of data is available in the utilization of the Weibull distribution, whereas a big volume of data is necessary in the utilization of the limit laws of extreme values, the main part being played by events with small probabilities at the "tail" of the distribution. In the usual case of the Weibull distribution when a relatively small number of data is used, the events with small probabilities appear seldom and the distribution curve corresponding to these data is very little different from that of a normal law or of gamma distributions with suitable parameters.

That is why before passing to the problem itself, the authors have considered it necessary to start by presenting the limit laws for the extreme values. It can be seen how it can be quite completely and rigorously presented by elementary means, without references (except for a lemma rather elementary of Feller [4], or better said Hincin-Feller). To this purpose, it was noticed that certain functional equations valid for values of n integer and positive numbers continue to be valid for r values, rationally positive numbers.

By resuming some of Cramer's results [5] it is pointed out among other things that for a certain asymptotic independence, it is not necessary to ask for the derivability of the distribution function. Certain shortcomings in the specialist literature are also outlined.

1. *Extreme values, limit laws*

Let X be a random variable whose probabilistic law is given by the distribution function $F(x)$ and $x_1, x_2, ..., x_n$ the results of n independent proofs, or in other words, a selection of volume n. Of course, in a second selection of volume n other results can be obtained,let them be $x_1', x_2', ..., x_n'$ a.s.o. Instead of considering a single random variable and a selection of volume n, n independent random variables $X_1, X_2, ..., X_n$ can be taken, with the same distribution function $F(x)$ and consider a single proof $x_1, x_2..., x_n$ corresponding to these n variables.

The problem arises how will the extreme values of various proofs, that is the random variables $M_n = max\ [X_1, X_2, ..., X_n]$ and $m_n = min\ [X_1, X_2, ..., X_n]$ behave.

Their distribution functions can be easily calculated. For $M_n < x$ where x is a certain real number, it is necessary that each of the values taken by variables $X_1. X_2, ..., X_n$ be smaller than x and reciprocally if the values taken by variables $X_1, .X_2, ..., X_n$ are smaller than x, then of course also $M_n < x$. Consequently:

$$P[M_n < x] = P[X_1 < x,\ X_2 < x, ...,\ X_n < x] =$$

$$= P[X_1 < x] \cdot P[X_2 < x] ... P[X_n < x] = F(x) \cdot F(x) ... F(x) = F^n(x)$$

because the random variables $X_1, X_2, ..., X_n$ are independent.

The distribution function of M_n is thus $F^n(x)$.

To find out the distribution function of m_n, it should be acted somewhat differently, namely the probability of the event $m_n \geqslant x$ is calculated. Or, for

this inequality to take place, it is necessary that all values of $X_1, X_2, ..., X_n$ taken into consideration be $\geqslant x$ and reciprocally. Consequently:

$$P[m_n \geqslant x] = P[X_1 \geqslant x, X_2 \geqslant x, ..., X_n \geqslant x] =$$
$$= P[X_1 \geqslant x] \cdot P[X_2 \geqslant x] ... P[X_n \geqslant x] =$$
$$= [1 - F(x)] \cdot [1 - F(x)] ... [1 - F(x)] = [1 - F(x)]^n.$$

Thus, the distribution function of m is:

$$P[m_n < x] = 1 - P[m_n \geqslant x] = 1 - [1 - F(x)]^n$$

At the same time, it is necessary to study the way in which the limit laws of the random variables M_n and $m_n (n \to \infty)$ are presented.

If for any finite x, $0 < F(x) < 1$, then

$$\lim_{n \to \infty} P[M_n < x] = \lim_{n \to \infty} F^n(x) = 0$$

and

$$\lim_{n \to \infty} P[m_n < x] = 1 - \lim_{n \to \infty} [1 - F(x)]^n = 1.$$

In other words, in this case the probability for the highest value attained by variables $X_1, X_2, ..., X_n$ be smaller than a certain fixed number (no matter how high) is smaller than a determined number $\varepsilon > 0$ no matter how small, if n is sufficiently big and similarly, the probability for the lowest value attained by the same variables to be smaller than a certain determined number (no matter how small), is bigger than $1 - \varepsilon$ if n is sufficiently high.

An interesting case is that when the random variable X is assuming only values comprised between two finite numbers a and b $(a < b)$. Therefore, the case should also be considered when for $a < x < b$; $0 < F(x) < 1$; $F(a) = 0$; $F(b) = 1$. Of course, for $a < x < b$,

$$\lim_{n \to \infty} P[M_n < x] = 0, \lim_{n \to \infty} P[m_n < x] = 1$$

and

$$\lim_{n \to \infty} P[M_n \to b] = 1, \lim_{n \to \infty} P[M_n \to a) = 1.$$

Of course it is desirable that we should know more than, in case $n \to \infty$ it is to be expected that $M_n \to 0$ and $m_n \to 1$ in the first case and that $M_n \to b$, $m_n \to a$ in the second case. More exactly, because as far as the applications are concerned, a limit case interests as an approximation under more precise conditions of finite cases, the above results in themselves are very little satisfactory.

It would be desirable to find in the first case nondegenerated probabilistic limit laws for high values in absolute value of the random variable, conveniently standardized and, in the second case, for values approaching a and b.

As the second case can be brought to the first, by a simple transformation we shall deal with the first.

For this, one started by using a method which proved useful in various cases, namely, instead of a direct investigation of the limit law or laws corresponding to the random variable X, one passed to those corresponding to auxiliary variables easier to study, namely as regards M_n to the random variable U_n whose values are $u = n[1 - F(x)]$, and as regards m_n to the random variable V_n whose values are $v = n \cdot F(x)$, x representing a certain value taken by X.

It is noticed that for a given x, u and v depend on the volume of selection (n) to different volumes correspond different values for u and v and reciprocally for u and v determined, to different volumes (n) correspond different values for x.

The distribution functions of U_n and V_n are obtained immediately. For an u determined from $u = n[1 - F(x)]$ results $F(x) = 1 - \frac{u}{n}$ and as the distribution function of M_n is $F^n(x) = \left[1 - \frac{u}{n}\right]^n$, of U_n will be $1 - \left(1 - \frac{u}{n}\right)^n$, because when x increases, u decreases. Similarly from $v = nF(x)$ results $F(x) = \frac{v}{n}$ and as the distribution function of m_n is $1 - [1 - F(x)]^n$ of V_n will be $1 - \left(1 - \frac{v}{n}\right)^n$, because x and v are varying in the same direction. The limit laws for the random variables U_n and V_n are obtained immediately. They are:

$$\lim_{n\to\infty} P[U_n < u] = 1 - \lim_{n\to\infty}\left(1 - \frac{u}{n}\right)^n = 1 - e^{-u}$$

respectively

$$\lim_{n\to\infty} P[V_n < v] = 1 - \lim_{n\to\infty}\left(1 - \frac{v}{n}\right)^n = 1 - e^{-v}.$$

Moreover, for v and v upper bounded (the lower bound is obviously 0), U_n and V_n tend towards these bounds uniformly, the approximation order being $\frac{1}{n}$. It suffices to note to this purpose that $\lim_{n\to\infty} \frac{\left(1 - \frac{t}{n}\right)^n - e^{-t}}{\frac{1}{n}} = \frac{t^2}{2} e^{-t}$.

(For example, the Hospital rule can be applied for calculating $\lim_{x\to\infty} \frac{\left(1 - \frac{t}{x}\right)^x - e^{-x}}{\frac{1}{x}}$ x being a real variable).

In order to use these results, one must find the values of x for high values of n, u and v being determined, in the case of various probabilistic laws, when technically, this is possible without too big complications.

An instructive and useful example in this respect is that of the normal law. Supposing $m = 0$, $\sigma = 1$, the distribution function is $F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$ and, consequently:

$$u = n[1 - F(x)] = \frac{n}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-t^2/2}\,dt.$$

The asymptotic solution of this equation (x being the unknown value) for high values of n is obtained by noting that $\int_{x}^{\infty} e^{-t^2/2}\,dt = \int_{x}^{\infty} \frac{1}{t}\,(te^{-t^2/2})\,dt$. By integrating by parts, the equation becomes:

$$\frac{\sqrt{2\pi}\cdot u}{x} = \frac{1}{x}\,e^{-x^2/2}\left[1 + O\left(\frac{1}{x^2}\right)\right].$$

Supposing u bouned, we obtain

$$x = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}} - \frac{\ln u}{\sqrt{2\ln n}} + O\left(\frac{1}{\ln n}\right)$$

wherefrom it results that the last term converges in probability to 0.

In the above equality, the right side includes $(-\ln u)$ and not u and, thus there is no question of a linear relationship between x and u (the last term excepted which tends in probability to 0). The relation would be linear if instead of u, we would consider a new random variable $w = -\ln u$ (thus $u = e^{-w}$) whose limit distribution function is $e^{-e^{-w}}$ (not $1 - e^{-e^{-w}}$, which would be obtained by replacing in $1 - e^{-u}$, u with e^{-w}, because when u increases w decreases).

Passing to the general normal law with the mean m and the quadratic mean deviation σ, in the above equality, by replacing x by $\frac{x - m}{\sigma}$ and $(-\ln u)$ by w, we obtain:

$$x = m + \sigma\sqrt{2\ln n} - \sigma\frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}} + \frac{\sigma}{\sqrt{2\ln n}}\,w + O\left(\frac{1}{\ln n}\right).$$

As regards m_n considering relation $n \cdot F(y) = v$ . . .

$F(y) = \frac{v}{n}$ and taking into account the fact that for a v determined, the greater n in comparison with v, ever higher negative values in absolute value correspond to v, substitution $y = -z$ can be made. By taking into account that in case of the normal law $F(y)$ is a symmetrical and continuous function, that is:

$F(y) = F(-z) = 1 - F(z)$. ·. $nF(y) = n[1 - F(z)$ and equation $n\,[1 - - F(z)] = v$ is the same with that at the beginning, only z appears instead of x and v appears instead of u.

For $m = 0$, $\sigma = 1$ the solution is

$$y = -z = -\sqrt{2 \ln n} + \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}} + \frac{\ln v}{\sqrt{2 \ln n}} + O\left(\frac{1}{\ln n}\right)$$

and in the general case (m and σ certain):

$$y = m - \sigma\sqrt{2 \ln n} + \sigma\frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}} - \frac{\sigma}{\sqrt{2 \ln n}} w' + O\left(\frac{1}{\ln n}\right) \text{ where } w' \text{ is}$$

a new random variable whose distribution function is $e^{-e^{-w'}}$ (we wrote $w' = = - \ln v$).

On the basis of those stated above, one can calculate the mean values and the dispersions of M_n and m_n obtaining:

$$E(M_n) = m + \sigma\left[\sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi - 2C}{2\sqrt{2 \ln n}} + O\left(\frac{1}{\ln n}\right)\right].$$

$$E(m_n) = m - \sigma\left[\sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi - 2C}{2\sqrt{2 \ln n}} + O\left(\frac{1}{\ln n}\right)\right]$$

where C is Euler's constant value: $C = \lim\limits_{n\to\infty}\left[\sum\limits_{k=1}^{n-1}\frac{1}{k} - \ln n\right] = 0.5772156...$

and:

$$D^2(M_n) = D^2(m_n) = \frac{\sigma^2}{2 \ln n} \cdot \frac{\pi^2}{6} + O\left(\frac{1}{\ln^2 n}\right).$$

Observations. 1) In the case of the normal law as well, it is true that $M_n \to \to +\infty$ and $m_n \to -\infty$ when $n \to \infty$, but $E(M_n)$ respectively $E(m_n)$ tend towards $+\infty$, respectively $-\infty$, in essence like $\sqrt{2 \ln n}$ respectively $-\sqrt{2 \ln n}$, thus rather slowly. Although one can speak about a concentration of the observation data (of the values of M_n respectively m_n) around the mean values, as dispersion tends to 0 when $n \to \infty$, this takes place much slower than in current practice, the dispersion being of the order $\frac{1}{\ln n}$ and not $\frac{1}{n}$.

2) Although, M_n and m_n are, of course, not independent, the same observation data being included into the calculus of the values of each of them, yet the random variables U_n and V_n are asymptotically independent and this may often serve in a useful estimation of mean values into which M_n and m_n are occuring simultaneously.

The above mentioned property can be established, for example, in the following way:

It is noticed that:

$$P(M_n < x,\ m_n < y] = P[M_n < x] - P[y \leqslant X_1,\ X_2, ..., X_n < x] =$$
$$= P[M_n < x] - P[y \leqslant X_1, < x] \cdot P[y \leqslant x_2 < x], ..., P[y \leqslant X_n < x] =$$
$$= F^n(x) - [F(x) - F(y)]^n$$

As

$$F(x) = 1 - \frac{u}{n},\ F(y) = \frac{v}{n}$$

it results that:

$$P[M_n < x,\ m_n < y] = \left(1 - \frac{u}{n}\right)^n - \left(1 - \frac{u+v}{n}\right)^n.$$

But:

$$P[M_n < x,\ m_n < y] = P[U_n \geqslant u,\ V_n < v] = P[V_n < v] - P[U_n < u, V_n < v];$$
$$P[U_n < u,\ V_n < v] = P[V_n < v] - P[U_n \geqslant u,\ V_n < v] =$$
$$= 1 - \left(1 - \frac{v}{n}\right)^n - \left(1 - \frac{u}{n}\right)^n + \left(1 - \frac{u+v}{n}\right)^n.$$

Consequently:

$$\lim_{n\to\infty}[U_n < u,\ V_n < v] = 1 - \lim_{n\to\infty}\left(1 - \frac{v}{n}\right)^n - \lim_{n\to\infty}\left(1 - \frac{u}{n}\right)^n +$$
$$+ \lim_{n\to\infty}\left(1 - \frac{u+v}{n}\right)^n = 1 - e^{-v} - e^{-u} + e^{-(u+v)} = (1 - e^{-u}) \cdot (1 - e^{-v}) =$$
$$= \lim_{n\to\infty} P(U_n < u) \cdot \lim_{n\to\infty} P(V_n < v).$$

thus the random variables U_n, V_n are asymptotically independent.

This result may serve in the case of the normal law to see to what extent the method of estimating the mean value by the semisum of the extreme values of selection is justified and how can the amplitude of selection be used for estimating the quadratic mean deviation.

Thus, noting that the values x and y obtained for M_n and m_n are characterized by their symmetry arround m, we conclude that to each term of x which contains u in a form or another correspond in y a similar term which contain v with the same coefficient but with oposite sign. As the distribution functions of U_n and V_n are the same, it is clear that the mean value of $\frac{M_n + m_n}{2}$ is m. Thus $\frac{x+y}{2}$ is a unbiased estimation of m.

If things are very simple as regards the mean value, we cannot say the same about dispersion, variables M_n and m_n not being independent,

but as U_n and V_n are asymptotically independent, it results from those shown above that

$$D^2\left(\frac{M_n + m_n}{2}\right) = \frac{2D^2(M_n)}{4} + o\left(\frac{1}{n}\right) = \frac{\sigma^2}{4 \ln n} \cdot \frac{\pi^2}{6} + O\left(\frac{1}{\ln^2 n}\right).$$

The selection mean $\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$ has the mean value m and the dispersion $\frac{\sigma^2}{n}$.

Thus, by considering instead of the selection mean, the mean of extremes $\left(\frac{x+y}{2}\right)$ umbiased estimation is similarly obtained but much of precision is lost, the dispersion being of the order $\frac{1}{\ln n}$ in comparison with $\frac{1}{n}$ at the selection mean.

As regards the estimation of the quadratic mean deviation with the aid of the extreme values it is noticed that

$$x - y = 2\sigma\left[\sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi - w - w'}{2\sqrt{2 \ln n}} + O\left(\frac{1}{\ln n}\right)\right]$$

and that

$$E(M_n - m_n) = 2\sigma\left[\sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi - 2C}{2\sqrt{2 \ln n}} + O\left(\frac{1}{\ln n}\right)\right];$$

$$D^2(M_n - m_n) = \frac{\sigma^2}{2 \ln n} \cdot \frac{\pi^2}{6} + O\left(\frac{1}{\ln^2 n}\right).$$

So, it is natural to denote $k_n = 2\sqrt{2 \ln n}\left[1 - \frac{\ln \ln n + \ln 4\pi - 2C}{8 \ln n}\right]$ and consider as an estimation of σ: $\frac{x - y}{k_n}$. Thus, the mean value and the corresponding dispersion are:

$$E\left(\frac{M_n - m_n}{k_n}\right) = \sigma\left[1 + O\left(\frac{1}{\ln^{3/2} n}\right)\right]$$

$$D^2\left(\frac{M_n - m_n}{k_n}\right) = \frac{\sigma^2}{16 \ln^2 n}\left[1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right].$$

The above estimation is consistent but biased, the shift being the smaller, as n is bigger. However, it must be noted that this estimation is less satisfactory than the usual one, namely:

$$\sqrt{m_2} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

because

$$E(\sqrt{m_2}) = \sigma\left[1 + O\left(\frac{1}{n}\right)\right], \text{ and } D^2(\sqrt{m_2}) = \frac{\sigma^2}{2n} + O\left(\frac{1}{n^2}\right).$$

Also within the normal law, it is necessary to find limit laws corresponding to the random variables M_n and m_n or better said to some linear functions (the coefficients depending on n) which should lead to nondegenerated limit laws.

For this, we use the expressions obtained for x and y. To ease the writing of the formulas we shall denote:

$$u'_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}}; \; \alpha'_n = \sqrt{2 \ln n}.$$

Resulting from the expression obtained for x is:

$$\alpha'_n\left(\frac{x - m - \sigma u'_n}{\sigma}\right) = w + O\left(\frac{1}{\sqrt{\ln n}}\right).$$

Thus:

$$\lim_{n\to\infty} P\left[\alpha'_n\left(\frac{M_n - m - \sigma u'_n}{\sigma}\right) < t\right] = \lim_{n\to\infty} P\left[w + O\left(\frac{1}{\sqrt{\ln n}}\right) < t\right] =$$

$$= [P\, w \leqslant t] = e^{-e^{-t}},$$

and the same from the expression of y is:

$$\alpha'_n\left(\frac{y - m + \sigma u'_n}{\sigma}\right) = -w' + O\left(\frac{1}{\sqrt{\ln n}}\right).$$

thus:

$$\lim_{n\to\infty} P\left[\alpha'_n\left(\frac{m_n - m + \sigma u'_n}{\sigma}\right) < t\right] = \lim_{n\to\infty} P\left[-w' + O\left(\frac{1}{\sqrt{\ln n}}\right) < t\right] =$$

$$= P[-w' \leqslant t] = P[w' \geqslant -t] = 1 - P[w' < -t] = 1 - e^{-e^{t}}.$$

Thus, the indirect method suggested by Cramer proves quite fruitful. For results also within other probabilistic laws see [5. chap. 28.6] where the somewhat more general case of some quantiles is studied.

It should be noticed that the above limit (bound) laws are sometimes presented insufficiently correct. See, for example [1], the formulas 5.3.149 and 5.3.150 as well as 5.3.147 used in the expression of the two formulas we refer to. Other times precision leaves much to be desired as in [6], point 14.18, the example 14.3.

In conclusion of this paragraph we should mention that we have insisted upon the normal law considering its importance in the problem which interests us and because we could clear up several important points and, at the same time, because it was seen that the limit distributions of the extremes are not

Weibull distributions. The basic problem namely the finding of distributions for which the conveniently standardized extremes admit nondegenerated limit distributions and the deduction of the various classes of limit distributions has remained open.

2. *The general case approach*

As results from the last part of the previous paragraph, although M_n and m_n have degenerated limit distributions, however in the case of the normal law, there are two sequence of numbers $\{a_n\}$ and $\{b_n\}$ so that $\frac{M_n - b_n}{a_n}$ should admit a nondegenerated limit distribution $\left(a_n = \frac{\sigma}{\alpha'_n};\ b_n = m + \sigma u'_n;\right.$ $\left.\alpha'_n;\ u'_n \text{ being defined there}\right)$. Similarly as regards m_n, there are two sequences $\{a'_n\}$ and $\{b'_n\}$ so that $\frac{m_n - b'_n}{a'_n}$ should admit a proper limit distribution $\left(a'_n = \frac{\sigma}{\alpha'_n};\ b'_n = m - \sigma u'_n\right)$.

So, it is natural that in the general case, the problem should be put as follows: considering the random variables X with the distribution function $F(x)$, there are two sequences of real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ so that $\frac{M_n - \beta_n}{\alpha_n}$ should admit a proper limit distribution, respectively two sequences of real numbers $\{\alpha'_n\}$, $\{\beta'_n\}$ so that $\frac{m_n - \beta'_n}{\alpha'_n}$ should admit a proper limit distribution? What can be said about the proper limit distributions when such distributions exist?

Obviously:

$$P\left[\frac{M_n - \beta_n}{\alpha_n} < x\right] = P[M_n < \alpha_n x + \beta_n] = F^n(\alpha_n x + \beta_n)$$

and

$$P\left[\frac{m_n - \beta'_n}{\alpha'_n} < x\right] = P[m_n < \alpha'_n x + \beta'_n] = 1 - [1 - F(\alpha'_n x + \beta'_n]^n.$$

Thus, it results that the cases should be studied in which the sequences of distribution functions $\{F^n(\alpha_n x + \beta_n)\}$ tends towards a nondegenerated distribution function and similarly as regards the sequences $\{1 - [1 - F(\alpha'_n x + + \beta'_n)]^n\}$.

We shall concentrate on the possible limit laws. To this purpose we shall start from a Feller's lemma which is applied to the sequences of distributions

functions with a much more general character ([4], VIII, 2 lemma 1) namely:

Let U and V be two probability distributions, none of them being concentrated at one point. If for a sequence of probability distributions $\{F_n\}$ and for the constraints $a_n > 0$ and $\alpha_n > 0$ we have

$$F_n(a_n x + b_n) \to U(x);\ F_n(\alpha_n x + \beta_n) \to V(x) \tag{1}$$

in all the continuity points then:

$$\frac{\alpha_n}{a_n} \to A \neq 0; \qquad \frac{\beta_n - b_n}{a_n} \to B \tag{2}$$

and

$$V(x) = U(Ax + B) \quad \text{is true.}$$

Conversely, if (2) takes place, then each of relations (1) implies the other.

The proof given by Feller is elementary, simple and ingenious.

In the first case which interests us $F_n = F^n$. Let us suppose that there are two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ so that $F^n(a_n x + b_n) \to U(x)$ in any continuity point of U.

The above sequence being convergent, any subsequence of it will be convergent and will tend to the same limit. In particular, if k is a certain natural number:

$$F^{kn}(a_{kn} + b_{kn}) \to U(x).$$

On the other hand:

$$F^n(a_n x + b_n) \to U(x) \Rightarrow [F^n(a_n x + b_n)]^k \to U^k(x) \cdot \cdot \cdot F^{kn}(a_n x + b_n) \to U^k(x).$$

We are in the case of the above mentioned lemma, namely in a particular case in which F^{kn} appears instead of F_n; a_{kn}, b_{kn} appear instead of a_n, b_n; a_n, b_n appear instead of α_n, β_n and $U^k(x)$ appears instead of $V(x)$.

According to it; $\frac{a_n}{a_{kn}} \to A_k$; $\frac{b_n - b_{kn}}{a_{kn}} \to B_k$. (We wrote A_k, B_k instead of A, B) and

$$U^k(x) = U(A_k x + B_k).$$

That is why and how the functional equation appears or better said the sequences of functional equations characterizing the limit functions and asymptotic relation existing between coefficients a_n and a_{kn} respectively between b_n, b_{kn} and a_{kn} and those of the corresponding functional equation.

How far we are from the too little satisfactory presentations in[1] and [2].

We presented above the functional equation corresponding to the maximum. In case of the minimum it is similarly shown that the functional equation is:

$$[1 - U'(x)]^k = 1 - U'(A'_k x + B'_k).$$

For solving the functional equations $U^k(x) = U(A_k x + B_k)$ it should be noticed that resulting from $U^{mn}(x) = [U^m(x)]^n = [U^n(x)]^m$ is:

$$U(A_{mn}x + B_{mn}) = U^n(A_m x + B_m) = U^m(A_n x + B_n)$$

that is:

$$U(A_{mn}x + B_{mn}) = U[(A_n(A_m x + B_m) + B_n] = U[(A_m(A_n x + B_n) + B_m]$$

The equations being valid in all the continuity points of U, it results that

$$A_{mn} = A_m A_n,\ B_{mn} = A_n B_m + B_n = A_m B_n + B_m$$

The last equality can also be written:

$$B_m(A_n - 1) = B_n(A_m - 1).$$

For $m = 1$, naturally $A_1 = 1$; $B_1 = 0$; $(U(x) = U(A_1 x + B_1) \to A_1 = 1$; $B_1 = 0)$.

If for a certain $m > 1$, $A_m = 1$, then, necessarily for any other $n > 1$, $A_n = 1$ because it is impossible that too $B_m = 0$ (thus $U^m(x) = U(x)$ would result and U would be degenerated). Consequently, we have a first case in which for any $n > 1$, $A_n = 1$ and B_k must satisfy equation: $B_{mn} = B_m + B_n$.

Let us suppose $A_2 \neq 1$, then $B_2(A_n - 1) = B_n(A_2 - 1) \Rightarrow \dfrac{B_n}{A_n - 1} = \dfrac{B_2}{A_2 - 1}$. If $B_2 = 0$ then also for any $n > 2$; $B_n = 0$ Finally, if $B_2 \neq 0$, we denote $\dfrac{B_2}{A_2 - 1} = \alpha$ Then $B_n = \alpha\,(A_n - 1)$.

In conclusion, the functional equation appears in one of the following three forms:

$$U^n(x) = U(x + B_n) \tag{3}$$

$$U^n(x) = U(A_n x) \tag{4}$$

$$U^n(x) = U(A_n(x + \alpha) - \alpha);\ \ \alpha \neq 0. \tag{5}$$

The functional equations (4) and (5) can be included into a single one, namely:

$$U^n(x) = U(A_n(x + \alpha) - \alpha);\ \alpha - \text{real} - (\text{therefore also } \alpha = 0). \tag{4'}$$

Consequently, one must examine the functional equations (3) and (4') with $n = 2,3, \ldots$ For this, we shall show that they are valid not only for natural number but also for n rationally, positive and then we shall find A_n and B_n as functions of n.

If relation (1) is true then also:

$$[F^n(a_n x + b_n)]^{1/q} = F^{n/q}(a_n x + b_n) \to U^{1/q}(x)$$

where q is a certain natural number.

As each subsequence of the sequences $\{F^{n/q}(a_n x + b_n)\}$ tends towards the same limit, then:

$$F^{qn/q}(a_{qn}x + b_{qn}) = F^n(a_{qn}x + b_{qn}) \to U^{1/q}(x).$$

Thus

$$F^n(a_n x + b_n) \to U(x);\ F^n(a_{qn}x + b_{qn}) \to U^{1/q}(x).$$

According to Feller's lemma it follows that:

$$U^{1\,q}(x) = U(A_{1/q}x + B_{1/q})$$

and

$$\frac{a_{qn}}{a_n} \to A_{1/q};\quad \frac{b_{qn} - b_n}{a_n} \to B_{1/q}.$$

Let p be a certain rationnally positive number. It must be noticed that:

$$[U^{1/q}(x)]^p = U^{p/q}(x) = U^p[A_{1/q}x + B_{1/q}] = U(A_p(A_{1/q}x + B_{1/q}) + B_p).$$

that is:

$$U^{p/q}(x) = U(A_{p/q}x + B_{p/q})$$

if we denote:

$$A_{p/q} = A_p A_{1/q};\quad B_{p/q} = A_p \cdot B_{1/1} + B_p.$$

We have thus proved that the functional equation $U^n(x) = U(A_n x + B_n)$ is valid not only for any n natural number but also for any r rationally positive number $(r = p/q)$.

Let us now proceed to finding A_n and B_m as functions of n and more generally of A_r and B_r as functions of r.

It was seen that $A_{mn} = A_m \cdot A_n$ and although the proof was made for m, n entirely positive, it is also valid for m, n rationally positive, because the functional equation is $U^r(x) = U(A_r x + B_r)$. Denoting $A_r = f(r)$ thus $A_s = f(s)$; $A_{rs} = f(rs)$; r, s being two certain rationally positive numbers, relation $A_{rs} = A_r A_s$ is transposed into equation

$$f(r\,s) = f(r) \cdot f(s). \tag{6}$$

To solve it, it is sufficient that f be defined only on the set of the rationally positive numbers and be subject to the single demand that there should be two rationally positive numbers r_1, r_2; $r_1 < r_2$ so that f should be lowerr or upper bounded by a real positive number on the interval $[r_1\ r_2]$. As it will be seen, this demand is satisfied in the problem which interests us.

Thus, let us admit that there exists a certain interval $[r_1,\ r_2]$; $0 < r_1 < r_2 < +\infty$ so that on this interval f be lower or upper bounded by a finite number $k > 0$ (or be upper bounded by such a number). In this case the only possible solution is $f(r) = r^\alpha$; α real number.

To justify those said, we shall notice that $f_0(r) = r^{-\alpha}$ with α a certain real number, is a solution of equation $f(rs) = f(r)f(s)$ and if $f(r)$ satisfies the above equation, then also $F(r) = r^{-\alpha}f(r)$ satisfies equation $F(rs) = F(r) \cdot F(s)$ as $F(rs) = (rs)^{-\alpha}f(rs) = (rs)^{-\alpha}f(r)\,f(s) = r^{-\alpha}f(r)\,s^{-\alpha}f(s) = F(r)F(s)$.

Let us denote $r_0 = \frac{r_2}{r_1}$ (naturally $r_0 > 1$) and choose α so that $F(r_0) = r_0^{-\alpha}f(r_0) = 1$ thus $\alpha = \frac{\ln f(r_0)}{\ln r_0}$. We shall now prove that by taking α in this way, $F(r) = 1$ for any r if the above demand is satisfied and thus $r^{-\alpha}f(r) = 1 \;\therefore\; f(r) = r^{\alpha}$. Indeed, from $F(r_0) = 1$ it results that also $F(r_0^2) = F(r_0) \cdot F(r_0) = 1$ and generally, that $F(r_0^n) = 1$, for n integer (positive or negative). ($F(1^2) = F(1) \cdot F(1) \Rightarrow F(1) = 1$ then

$$F(r_0 r_0^{-1}) = F(r_0) \cdot F(r_0^{-1}) \Rightarrow F(r_0^{-1}) = \frac{1}{F(r_0)} = 1$$

and thus $F((r_0^{-1})^n) = 1$; $F(r_0^{-n}) = 1$; n is a positive integer. Let us show that $F(r) = 1$ for any r rationally positive number. Let us suppose that it would not be so. Then a positive rational number R would exists, such that $F(R) \neq 1$. To admit that $F(R) > 1$ (otherwise instead of R we would consider $\frac{1}{R}$ because $f\left(\frac{1}{R}\right) = \frac{1}{f(R)}$ and if $f(R) < 1$; $f\left(\frac{1}{R}\right) > 1$). By starting from $F(R) = c > 1$; and as $F(R^p) = F^p(R) = c^p$, an integer number p can be chosen (negative or positive) so that $F(R^p) < k$ (respectively $F(R^p) > k$ if f is upper bounded). Or $F(r_0^n R^p) = F(r_0^n) \cdot F(R^p) = F(R^p) < k$ for some n integer number.

The sequence $\{r_0^n R^p\}$ is increasing ($-\infty < n < +\infty$) by terms no matter how small or high ($\lim_{n \to -\infty} r_0^n = 0$; $\lim_{n \to -\infty} r_0^n = +\infty$). Let $r_0^s S^p$ be the highest term of this sequence smaller than r_1; then the following term $r_0^{s+1}R^p$ will necessarily be included into the interval $[r_1, r_2]$ because otherwise $r_0^s R^p < r_1 < r$, $< r_0^{s+1}R^p$ and thus $\frac{r_0^{s+1}R^p}{r_0^s R^p} > \frac{r_2}{r_1} \;\therefore\; r_0 > \frac{r_2}{r_1}$ which is impossible because $r_0 = \frac{r_2}{r_1}$. Consequently $r_0^{s+1}R^p$ would fall in the interval $[r_1, r_2]$ and $F(r_0^{s+1}R^p) = F(R^p) < k$ contrary to the hypothesis, because we supposed $F(r) > k$ for any $r \in [r_1, r_2]$. Thus, it is impossible that there should be an R with $F(R) \neq 1$ thus $F(r) = 1$ for any r rationally positive. Thus, the property is proved ($A_r = r^{\alpha}$). or otherwise written $A_r = r^{\gamma}$.

The proof would have been absolutely the same if instead of supposing $f(r) > k$ for any $r \in [r_1, r_2]$, we would have supposed $f(r) < k$ only, here number p would have been considered so that $f(R^p) > k$.

In order to find out B_r starting from the relation $B_{rs} = B_r + B_s$; where r, s are rational positive numbers (functional equation 3). Denoting by $B_r = f(r)$ the above relation can be written

$$f(rs) = f(r) + f(s).$$

In this case it must be noted that $f(r)$ may take both positive and negative values. If we take $r = s = 1$ it is seen that $f(1) = 0$ and $f(r) + f\left(\frac{1}{r}\right) = f\left(r\frac{1}{r}\right) = f(1) = 0$. $\therefore$ $f\left(\frac{1}{r}\right) = -f(r)$. Therefore condition $f(r) > k'$ refers to k' negative and the alternative $f(r) < k''$ to k'' positive.

But $f_0(r) = \lambda \ln r$, and $F(r) = f(r) - \lambda \ln r$; λ is taken so that $F(r_0) = 0$ and as $F(r_0^n) = nF(r_0) = 0$; $F(r_0^n r) = F(r_0^n) + F(r) = F(r)$. If $F(R) \neq 0$ then as $F(R^p) = pF(R)$ the integer number p(negative or positive) may be taken so that $pF(R) < k'$ (respectively $pF(R) > k''$). The rest of the proof is the same as above. So, in this case the general solution is $f(r) = \lambda \ln r$. It is very important that nondegenerated limit distributions U, for extreme values exist only if the conditions of the above type are satisfied. Indeed, if the equation is $U^r(x) = U[A_r(x + \alpha) - \alpha]$. as for $r' < r''$ and those x with $U(x) \neq 0$ or 1; $U^{r'}(x) > U^{r''}(x)$ thus $U[A_{r'}(x + \alpha) - \alpha] > U[A_{r''}(x + \alpha) - \alpha] \Rightarrow A_{r'}(x + \alpha) - \alpha > A_{r''}(x + \alpha) - \alpha \Rightarrow A_{r'} > A_{r''}$ if $x + \alpha > 0$ respectively $A_{r'} < A_{r''}$ if $x + \alpha < 0$; in other words $f(r') > f(r'')$respectively $f(r') < f(r'')$ if $r' < r''$. Function f is monotonous, thus on a certain interval $[r_1, r_2]$, the maximum and minimum (positive numbers) are at the extreme ends, thus they are finite numbers ($f(r_1)$ or $f(r_2)$). The conditions of our functional equation are fulfilled to excess. Similarly if the equation for the maximum is of the form $U^r(x) = U(x + B_r)$; then if $r' < r''$; $B_{r'} > B_{r''}$ and the conclusions are the same, but the minimum respectively the maximum on a given interval may be negative numbers as well. So, $B_r = \lambda \ln r$. or otherwise written $B_r = \gamma \ln r$.

It results from those shown above that the functional equations corresponding to the limit distributions of the maximum are:

$$U^r(x) = U(x + \gamma \ln r) \tag{3'}$$

$$U^r(x) = U(r^\gamma(x + \alpha) - \alpha) \tag{4''}$$

for any r rationally positive.

3. *The limit distributions of the maximum*

We shall start with equation 3'. In the first place, we must show which is the sign of γ. As $U^{r+1}(x) = U^r(x) \cdot U(x) < U^r(x)$ it results that $U(x + \gamma \ln (r + 1)) < U(x + \gamma \ln r)$ and thus $x + \gamma \ln (r + 1) < x + \gamma \ln r$. $\therefore$ $\gamma \ln \frac{r+1}{r} < 0$ and thus $\gamma < 0$.

Therefore, instead of γ we shall write $-\gamma$, $\gamma > 0$.

Thus, the equation is $U^r(x) = U(x - \gamma \ln r)$. for $\gamma > 0$.

By using the method in [5, §5.1.2] we shall solve this equation with complete proofs as here r is a rational number. By logarithmizing twice the above equality we obtain:

$$\ln r + \ln (-\ln U(x)) = \ln (-\ln U(x - \gamma \ln r)$$

Consequently:

$$U(x) = e^{-\left(-\frac{x+\alpha}{c}\right)^{1/\gamma}} \qquad \text{for } x < -\alpha;\ U(x) = 1 \text{ for } x \geqslant -\alpha.$$

The limit distribution contains three parameters α, c, γ of which c and γ positive.

If instead of the random variables X with the above distribution function we consider the random variable $Y = \dfrac{X+\alpha}{c}$, its distribution function will be:

$$U(y) = e^{-(-y)^{\beta}}, \qquad \text{for } y < 0 \text{ and } U(y) = 1 \text{ for } y \geqslant 0,$$

where $\beta = \dfrac{1}{\gamma} > 0$.

This distribution appears as a particular case of the above one, with $\alpha = 0$, $c = 1$, $\dfrac{1}{\gamma} = \beta$.

4. *The limit distribution of the minimum.*

As it was shown in § 2, the functional equation in the case of the minimum is $[1 - U'(x)]^k = 1 - U'(A'_k x + B'_k)$ valid for any continuity point x of $U'(x)$ and any natural number k.

Above, whenever we said any x, we implied any continuity point of $U(x)$.

We return to this equation in order to extend it from k natural number to r rationally positive number.

Also referring to Feller's lemma we notice that here $F_n(x) = 1 - (1 - F(x))^n$ and so if $1 - (1 - F(a'_n x + b'_n))^n \to U'(x)$ that is $[1 - F(a'_n x + b'_n)]^n \to 1 - U'(x)$ then.

$$[1 - F(a'_n x + b'_n)^n]^{1/q} = [(1 - F(a'_n x + b'_n)]^{n/q} \to (1 - U'(x))^{1/q}$$

for q a certain natural number and thus:

$$[1 - F(a'_{qn} x + b'_{qn})]^{qn/q} = [1 - F(a'_{qn} x + b'_{qn})]^n \to [1 - U'(x)]^{1/q}.$$

Thus:

$$F_n(a'_n x + b'_n) = 1 - F[(a'_n x + b'_n)]^n \to 1 - (1 - U'(x)) = U'(x)$$

$$F_n(a'_q x + b'_{qn}) = 1 - (1 - F(a'_{qn}\, x + b'_{qn}))^n \to 1 - (1 - U'(x))^{1/q}.$$

According to Feller's lemma:

$$1 - (1 - U'(x))^{1/q} = U'(A'_{1/q} x + B'_{1/q}) \cdot \cdot [1 - U'(x)]^{1/q} = 1 - U'(A'_{1/q} + B'_{1/q}).$$

Equally as in the previous paragraph:

$$[(1 - U'(x))^{1/q}]^p = [1 - U'(x)]^{p/q} = [1 - U'(A'_{1/q} x + B'_{1/q})]^p =$$
$$= 1 - U'(A'_{p/q} x + B'_{p/q})$$

thus $\left(\frac{r+1}{r}\right)^{\gamma} > 1$ (because $x + \alpha < 0$). It results that $\gamma > 0$.

To solve the equation, we logarithmize twice and obtain:

$$\ln r + \ln (-\ln U(x)) = \ln [-\ln U(r^{\gamma}(x+\alpha) - \alpha)].$$

By denoting $\ln (-\ln U(x)) = \Phi(x)$, the equation becomes:

$$\ln r + \Phi(x) = \Phi[r^{\gamma}(x+\alpha) - \alpha].$$

We seek a particular solution, but, this time of the form $a \ln [-(x+\alpha)]$ because $(x+\alpha) < 0$. By replacing, it is seen that such a solution exists for $a = \frac{1}{\gamma}$.

Thus, we obtained the particular solution $\Phi_0(x) = \frac{1}{\gamma} \ln [-(x+\alpha)]$. Denoting $\psi(x) = \Phi(x) - \Phi_0(x) = \Phi(x) - \frac{1}{\gamma} \ln [-(x+\alpha)]$ that is $\Phi(x) = \psi(x) + \frac{1}{\gamma} \ln [-(x+\alpha)]$ and by replacing in the above equation, we obtain:

$$\psi(x) = \psi[r^{\gamma}(x+\alpha) - \alpha]$$

an equation which must be satisfied for any $x < -\alpha$ and any r rationally positive. It must be proved that under these conditions $\psi(x)$ is constant. For this, a sequence of r rational numbers is considered, so that $r^{\gamma}(x+\alpha) - \alpha \to a$, x being certain: $(x < -\alpha)$ and a a continuity point of ψ $(a < -\alpha)$. Thus it is required that $r^{\gamma} \to \frac{a+\alpha}{x+\alpha}$ therefore $r \to \left(\frac{a+\alpha}{x+\alpha}\right)^{1/\gamma}$.

It must be noticed that as both $a + \alpha < 0$ and $(x+\alpha) < 0$ implies is $\frac{a+\alpha}{x+\alpha} > 0$. Let $\{r_n\}$ be such a sequence of rational numbers. As $\psi(x) = \psi[r_n^{\gamma}(x+\alpha) - \alpha]$ for any n it results that:

$$\psi(x) = \lim_{n \to \infty} \psi[r_n^{\gamma}(x+\alpha) - \alpha = \psi(a).$$

It was thus proved that for any $x < -\alpha$; $\psi(x) = \psi(a)$. Let us denote for the convenience of what will follow, this constant value by $-\frac{1}{\gamma} \ln c$ (of course $c > 0$). Then:

$$\Phi(x) = \psi(x) + \frac{1}{\gamma} \ln [-(x+\alpha) \cdot \cdot \cdot \Phi(x) = \frac{1}{\gamma} \ln [-(x+\alpha) - \frac{1}{\gamma} \ln c =$$

$$= \ln \left(-\frac{x+\alpha}{c}\right)^{1/\gamma}$$

and thus:

$$\ln (-\ln U(x)) = \ln \left(-\frac{x+\alpha}{c}\right)^{1/\gamma}; \; x < -\alpha.$$

the equation which must be satisfied for any $x > -\alpha$ and any r rationally positive. Let x be certain ($x > -\alpha$) and a continuity point of $\psi(a > -\alpha)$. We shall now consider a sequence of r values so that $\frac{x+\alpha}{r^\gamma} - \alpha \to a$ which implies $r \to \left(\frac{a+\alpha}{x+\alpha}\right)^{1/\gamma}$. Let $\{r_n\}$ be this sequence then, as $\psi(x) = \psi\left(\frac{x+\alpha}{r_n^\gamma} - \alpha\right)$ for any n, it follows that $\psi(x) = \lim_{n\to\infty} \psi\left(\frac{x+\alpha}{r_n^\gamma} - \alpha\right) = \psi(a)$. We have thus proved that for any $x > -\alpha$; $\psi(x) = \psi(a)$, thus $\psi(x)$ is constant for $x > -\alpha$. Let us denote $\psi(a) = \frac{1}{\gamma} \ln c$ (of course $c > 0$). As $\psi(x) = \Phi(x) + \frac{1}{\gamma} \ln(x+\alpha) \cdot \cdot \frac{1}{\gamma} \ln c =$ $= \Phi(x) + \frac{1}{\gamma} \ln(x+\alpha)$ for $x > -\alpha$. Consequently:

$$\Phi(x) = -\frac{1}{\gamma} \ln\left(\frac{x+\alpha}{c}\right) = \ln\left(\frac{x+\alpha}{c}\right)^{-1/\gamma}$$

so that:

$$\ln(-\ln U(x)) = \ln\left(\frac{x+\alpha}{c}\right)^{-1/\gamma}; \; x > -\alpha$$

an thus:

$$U(x) = e^{-\left(\frac{x+\alpha}{c}\right)^{-1/\gamma}} \qquad \text{for } x > -\alpha \text{ and}$$
$$U(x) = 0, \text{ for } x < -\alpha.$$

We have thus found the limit distribution for the first subcase in 4''. It contains three parameters α, γ, c of which $\gamma > 0$, $c > 0$.

If instead of the random variable X with the above distribution we consider variable $Y = \frac{X+\alpha}{c}$, the corresponding distribution function is:

$U(y) = e^{-\beta^{-y}}$ for $y > 0$ and $U(y) = 0$, for $y \leqslant 0$, $\beta = \frac{1}{\gamma} > 0$. This distribution appears as a particular case of the one above, namely that obtained by considering $\alpha = 0$, $c = 1$ and by denoting $\frac{1}{\gamma} = \beta$.

Let us now pass to the second subcase of 1'', namely to the solution of equation:

$$U^r(x) = U(r^\gamma(x+\alpha) - \alpha), \qquad \text{for } x < -\alpha,$$
$$U(x) = 1, \text{ for } x \geqslant -\alpha.$$

Here too, we start from inequality ($x < -\alpha$):

$$U^{r+1}(x) < U^r(x) \Rightarrow > U[(r+1)^\gamma(x+\alpha) - \alpha] < U[r^\gamma(x+\alpha) - \alpha] \Rightarrow$$
$$\Rightarrow (r+1)^\gamma(x+\alpha) - \alpha < r^\gamma(x+\alpha) - \alpha$$

Of course, if instead of a random variable X with the above distribution function we would take the random variable $Y = \frac{X - c}{\gamma}$ its distribution function will be $e^{-e^{-y}}$.

Let us now pass to case 4''. Here, the functional equation is:

$$U^r(x) = U(r^\gamma(x + \alpha) - \alpha).$$

Considering $r = 2$; $x = -\alpha$ we obtain

$$U^2(-\alpha) = U(-\alpha) \cdot \cdot U(-\alpha)[U(-\alpha) - 1] = 0.$$

Consequently, either $U(-\alpha) = 0$ or $U(-\alpha) = 1$. Hence, it follows that two subcases must be taken into account: in the first subcase, one must consider only the values x of the random variable higher or equal to $(-\alpha)$, in the second case only the values lower or equal to $(-\alpha)$.

Let us consider the first subcase $(x \geqslant -\alpha)$. If $x > -\alpha$ then as $U^{r+1}(x) < U^r(x)$ it results that $U[(r+1)^\gamma (x + \alpha) - \alpha] < U[r^\gamma(x + \alpha) - \alpha]$ and thus $(r+1)^\gamma (x + \alpha) - \alpha < r^\gamma(x + \alpha) - \alpha \Rightarrow \left(\frac{r+1}{r}\right)^\gamma < 1$ so that $\gamma < 0$.

Writing instead of γ, $-\gamma$, $\gamma > 0$, the functional equation becomes:

$$U^r(x) = U\left(\frac{x + \alpha}{r^\gamma} - \alpha\right).$$

By logarithmizing twice we obtain:

$$\ln r + \ln[-\ln U(x)] = \ln\left[-\ln\left(U\left(\frac{x + \alpha}{r^\gamma} - \alpha\right)\right)\right].$$

Denoting $\ln(-\ln U(x)] = \Phi(x)$ the above equation can be written:

$$\ln r + \Phi(x) = \Phi\left(\frac{x + \alpha}{r^\gamma} - \alpha\right).$$

The form of this equation suggests the finding of a particular solution of the form $a \ln(x + \alpha)$. By replacing, we obtain:

$$\ln r + a \ln(x + \alpha) = a \ln\left(\frac{x + \alpha}{r^\gamma}\right) \Rightarrow \ln r = -a\gamma \ln r;\ a = -\frac{1}{\gamma}.$$

Thus, we obtained the particular solution $\Phi_0(x) = -\frac{1}{\gamma} \ln(x + \alpha)$.

Denoting $\psi(x) = \Phi(x) - \Phi_0(x) = \Phi(x) \frac{1}{\gamma} \ln(x + \alpha)$ we obtain:

$$\psi(x) = \psi\left(\frac{x + \alpha}{r^\gamma} - \alpha\right)$$

Denoting $\Phi(x) = \ln(-\ln(U(x))$, the equation can also be written

$$\ln r + \Phi(x) = \Phi(x - \gamma \ln r).$$

An immediate solution is one of the form ax. By replacement we obtain:

$$\ln r + ax = a(x - \gamma \ln r) \Rightarrow a = -\frac{1}{\gamma}.$$

Consequently, a particular solution is $\Phi_0(x) = -\frac{1}{\gamma} x$.

Consider now $\psi(x) = \Phi(x) - \Phi_0(x) = \Phi(x) + \frac{1}{\gamma} x$, thus $\Phi(x) = \psi(x) - \frac{1}{\gamma} x$

By replacing in the equation concerning Φ we obtain equation

$$\psi(x) = \psi(x - \gamma \ln r)$$

valid for any x and any rational number r.

Let r be definite different from 1. The above equation shows that ψ it admits period $\gamma \ln r$. (Instead of x consider $x + \gamma \ln r$, then $x + 2\gamma \ln r$ etc. It can thus be seen that $\psi(x) = \psi(x + \gamma \ln r) = \psi(x + 2\gamma \ln r)$ etc.).

Similarly, consider instead of x, $x - \gamma \ln r$, respectively $x - 2\gamma \ln r$ etc. it is thus seen that

$$\psi(x) = \psi(x + k\gamma \ln r); \; k \pm 1, \pm 2, \ldots$$

But, in this case ψ is a function which would admit periods no matter how small because r can be considered no matter how close to 1, which is possible only if ψ is a constant value.

Indeed, let a be a continuity point of ψ (thus of U) and b another point so that $\psi(b) \neq \psi(a)$. Consider $|\psi(b) - \psi(a)| = \varepsilon$. In this case, no matter how close r would be to 1, $b = a + k\gamma \ln r + h$ with k conveniently chosen and with $|h| < \gamma \ln r$ (suppose $r > 1$). Consequently $\psi(b) = \psi(a + k\gamma \ln r + h) = \psi(a + h)$. As ψ is continuous in a, r can be taken sufficiently close to a so that $|\psi(a + h) - \psi(a)| < \varepsilon$ thus $|\psi(b) - \psi(a)| < \varepsilon$, that is $|\psi(b) - \psi(a)| < |\psi(b) - \psi(a)|$ We have come to a contradiction. Therefore $\psi(b) = \psi(a)$ as b was a certain value of x results $\psi(x) = \psi(a) = C$ for any x. But $\Phi(x) = \psi(x) - \frac{1}{\gamma} x$, thus $\Phi(x) = -\frac{1}{\gamma} x + C$. Consequently:

$$\ln(-\ln U(x)) = \Phi(x) = -\frac{1}{\gamma} x + C, \text{ Therefore, } U(x) = e^{-e^{-1/\gamma x + C}}.$$

We can write $C = \frac{1}{\gamma} c$ wherefrom:

$$U(x) = e^{-e^{-1/\gamma \cdot (x-c)}}.$$

We have thus found the limit law in case 3'. This depends on two parameters: $\gamma > 0$ and c.

where p is a certain natural number. Consequently, the functional equation to be considered:

$$(1 - U'(x))^r = 1 - U'(A'_r x + B'_r)$$

valid for any x, continuity point of $U'(x)$ and any r rationally positive.

If we denoted $1 - U'(x) = V'(-x)$, $V'-x)$ would be an distribution function because if $x_1 < x_2$: $V'(-x_1) = 1 - U'(x_1) \geqslant 1 - U'(x_2) = V'(-x_2)$; consequently $V'(-x_1) \geqslant V'(-x_2)$ for $-x_1 > -x_2$ and obviously $V'(-x)$ takes values between 0 and 1 similarly as $U'(x)$. But, if instead of $1 - U'(x)$ we write $V'(-x)$ the functional equation becomes:

$$V'^r(-x) = V'(-A'_r x + B'_r)$$

valid in any continuity point of V' and for any r rationally positive.

If a is the continuity point of U', $(-a)$ is a continuity point of V', because $1 - U'(x) = V'(-x)$, that is

$$\lim_{x \to a} [1 - U'(x)] = \lim_{(-x) \to (-a)} V'(-x).$$

By replacing $-x$ by y the functional equation becomes:

$$V'^r(y) = V'(A'_r y + B'_r).$$

Or this, in fact, is the very functional equation at paragraph 2, but the notations are different (V' appears instead of U), y instead of x, A'_r, B'_r instead of A_r, B_r so the solutions are the same, with the difference that x is replaced by y. The solutions are thus:

I. $V'(y) = e^{-e^{-1/\gamma \cdot (y-c)}}$; $\gamma > 0$, $-\infty < y < +\infty$.

II. $V'(y) = e^{-\left(\frac{y+\alpha}{c}\right)^{-1/\gamma}}$, for $y > -\alpha$;

$V'(y) = 0$, for $y \leqslant -\alpha$, γ, c positive.

III. $V'(y) = e^{-\left(-\frac{y+\alpha}{c}\right)^{1/\gamma}}$ for $y < -\alpha$;

$V'(y) = 1$, for $y > \alpha$, γ, c, pozitive.

As $U'(x) = 1 - V'(-x)$ the limit distribution corresponding to the minimum are:

I'. $U'(x) = 1 - e^{-1/\gamma \cdot (x+c)}$, $-\infty < x < +\infty$, $\gamma > 0$

II'. $U'(x) = 1 - e^{-\left(-\frac{x-\alpha}{c}\right)^{-1/\gamma}}$, for $x < \alpha$;

$U'(x) = 1$, for $x \geqslant \alpha$, γ, c positive;

III' $U'(x) = 1 - e^{-\left(\frac{x-\alpha}{c}\right)^{1/\gamma}}$, for $x > \alpha$

$U'(x) = 0$, for $x \leqslant \alpha$, γ, c positive.

The last limit distribution is called the Weibull distribution.

In conclusion, it should be pointed out that the above results were obtained in the hypothesis that the initial distribution function F is so that there is a limit distribution for M_n respectively m_n and it is not concentrated in a point. The problem has remained open of finding the conditions which have to be fulfilled by F for the existence of a limit distribution for M_n respectively m_n and thus, it should belong to one of the above three types. More precisely, one should establish the conditions which should lead to limit distributions of each type. We shall clear up this problem in a future article.

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LOIS-LIMITE POUR LES VALEURS EXTRÉMALES, LA RÉPARTITION WEIBULL ET QUELQUES PROBLÈMES STATISTIQUES DE L'ANALYSE GLOBALE DU CONTRÔLE

Cet ouvrage a pour point de départ certaines constatations concernant quelques opinions sur le caractère d'universalité qu'aurait la répartition Weibull et, par conséquant, le droit d'être utilisé à la solution des problèmes d'analyse globale du contrôle de réception. Le support théorique de ceux qui se rangent à cette opinion est la place à part que la répartition Weibull occupe dans la statistique d'ordre ou, pour mieux dire, entre les lois limite des valeurs extrémales

Comme d'ailleurs dans tant d'autres cas, il existe le risque — partant de certaines situations où l'emploi d'une répartition ou d'une autre est justifié — de passer à d'autres situations sans faire d'analyse préalable attentive et d'aboutir à des conclusions erronnées. Pour une juste compréhension des problèmes il est nécessaire de faire une analyse attentive des conditions assurant la valabilité des lois limite pour les valeurs extrémales et dans quelle mesure ces lois peuvent être appliquées lorsqu'il est question d'un nombre relativement petit de données. Or, les lois limite pour les valeurs extrémales ne sont point présentées du tout dans les monographies et les traités de statistique mathématique, ou bien elles sont présentées de façon incomplète et sans rigueur comme dans Kendall et Stuart (1) ou encore d'une manière succinte et imprécise comme dans la monographie de Gumbel sur les valeurs extrémales (2). D'habitude, lorsqu'il s'agit de lois limite concernant les valeurs extrémales on fait un renvoi à un mémoire célèbre de Gnedenko (3) peu accessible et difficile à suivre pour les non-spécialistes. D'ailleurs ce mémoire aussi fait un renvoi à un autre mémoire — celui de Hincin, encore plus difficile à trouver.

C'est pourquoi les auteurs de cet ouvrage essayent de présenter à ceux qui s'occupent des applications de la statistique mathématique dans l'économie les lois limites sous une forme accesible mais pourtant complète.

ПРЕДЕЛЬНЫЕ ЗАКОНЫ ДЛЯ ЭКСТРЕМАЛЬНЫХ ЗНАЧЕНИЙ, РАСПРЕДЕЛЕНИЕ ВЕЙБУЛА И НЕКОТОРЫЕ ВОПРОСЫ ГЛОБАЛЬНОГО АНАЛИЗА КОНТРОЛЯ

Исходной точкой данной работы являются определенные констатации относительно некоторых утверждений, касающихся универсальности характера, который якобы присущ распределению Вейбула, а следовательно и права быть использованным в решении вопросов глобального анализа контроля приема. Теоретической опорой тех, кто разделяет эту точку зрения является особое место, которое занимает распределение Вейбула в порядковой статистике, или лучше сказать среди предельными законами экстремальных значений.

Как и во многих других случаях здесь имеется риск того, что, исходя из определенного положения, при котором использование того или другого распределения оправдано, можно перейти к другим без предварительного внимательного анализа и сделать ошибочные выводы. Для правильного понимания вопросов необходим внимательный анализ условий, в которых действительны предельные законы для экстремальных значений и меры, в какой они могут быть применены в том случае, когда идет речь о сравнительно малом количестве данных. Но предельные законы для экстремальных значений либо не представлены вовсе в монографии и трактатах математической статистики, либо представлены со слишком большими оговорками и с недостаточной точностью как у Кендалла и Стюарта (I), либо очень кратко и с неточностью как в монографии Гумбела для экстремальных значений (2). Как правило, когда идет речь о предельных законах для экстремальных значений, ссылаются на известную работу Гнеденко, трудно доступную и совсем нелегкую для неспециалистов. Впрочем и эта работа ссылается на другую работу Хинчина, которую еще труднее найти.

Поэтому авторы стараются представить в этой работе предельные законы для экстремальных значений в доступной форме для тех, кто занимется применением математической статистики в экономике.

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