DOINA PETRONIU
University of Bucharest

Statistical Analysis of Multiplicative Models in the Case of a Log Transformation

The paper examines a class of multiplicative models which can be linearized by log transformation. For studying the parametric functions, the notion of function admitting a biased estimate is introduced and a theorem of the Gauss-Markov type is established in this case. On the basis of the properties of the parameters estimates and of the functions which can be estimated, confidence regions for the simultaneous estimation of such functions and tests are constructed. The paper solves an important problem for econometrics because the multiplicative models contain as a peculiar case the class of production functions of the COBB-DOUGLAS type.

1. INTRODUCTION

The examined multiplicative model is reduced to a linear model by log transformation. According to the aim pursued, the errors in the linear model may have a mean value different from zero and consequently the classical theory of linear models can no longer be applied. This is the case considered in the paper.

The parameters of the linear model are estimated by the least-squares method which no longer derives from the principle of maximum likelihood as in the classical theory.

In order to study the linear functions of the linear model parameters, the notion of function admitting a biased estimate is introduced.

The results obtained—confidence regions for sets of functions admitting a biased estimate and tests for verifying certain hypotheses on these functions-are transposed for the functions depending on the parameters of the multiplicative model by returning to the notations in the initial model.

2. THE MULTIPLICATIVE MODEL

It is supposed that n observations or measurements are known which will be denoted by $Y = (Y_1, ..., Y_n)'$*

Consider the multiplicative model:

$$Y_i = \delta_1 \prod_{j=2}^{p} Z_{ji}^{\delta_j} \cdot v_i \quad (i = 1, ..., n) \tag{1}$$

where $\delta = (\delta_1, ..., \delta_p)'$ is the vector of the unknown parameters, Z_{ji} are known positive numbers and $v = (v_1, ..., v_n)'$ is the random vector of the positively supposed disturbances.

There are several functions which can be studied by means of this model. An important example are the Cobb-Douglas type functions for production. Other such examples are given by Dhrymes [1], Laurent [3], Goldberger [2] Teekens and Koerts [5], etc.

By log transformation in (1), the multiplicative model is reduced to the linear model:

$$y = X'\beta + u \tag{2}$$

with

$$y = (ln\, Y_1, ..., ln\, Y_n)',\ X' = \begin{pmatrix} 1 & ln\, Z_{21} & \dots & ln\, Z_{p1} \\ \vdots & & & \\ 1 & ln\, Z_{2n} & \dots & ln\, Z_{pn} \end{pmatrix}$$

$$\beta = (ln\, \delta_1,\ \delta_2, \dots\ \delta_p)' \text{ and } u = (ln\, v_1, ..., ln v_n)'.$$

To simplify the writing, the components of the vectors and elements of the matrix in (2) will be renoted as follows:

$$y = (y_1, ..., y_n)',\quad X' = \begin{pmatrix} 1 & x_{21} & \dots & x_{p1} \\ \vdots & & & \\ 1 & x_{2n} & \dots & x_{pn} \end{pmatrix}$$

$$\beta = (\beta_1, ..., \beta_p)' \text{ and } u = (u_1, ..., u_n)'.$$

In the classical theory of linear models, the model (2) is studied under the fundamental hypothesis:

$$\Omega_1 : u \text{ is } N(0, \sigma^2 I)^{**}$$

where σ^2 is an unknown constant and I is the $n \times n$ identity matrix.

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*) The same symbols for the random variables and for their observed values will be used.

**) It will clearly result from the context when we speak of scalar 0 and of vector 0 which has all components nul.

Consequently, since $u_i = ln\, v_i$, $(i = 1, ..., n)$ and u_i is $N(0, \sigma^2)$, the original model (1) is examined under the fundamental hypothesis:
Ω_1^0: v has the lognormal distribution with

$$E(v) = e^{\frac{\sigma^2}{2}} \iota \text{ and } Cov(v, v') = (e^{2\sigma^2} - e^{\sigma^2})\, I$$

where we denote by ι the $n \times 1$ vector with all components equal to 1.

In this case it is noticed that:

$$E(Y_i) \neq \delta_1 \prod_{j=2}^{p} Z_{ji}^{\delta_j}$$

which contravenes our idea about the multiplicative models.

If in the linear model we consider instead of hypothesis Ω_1 the fundamental hypothesis:

$$\Omega_2: u \text{ is } N\left(-\frac{1}{2}\sigma_i^2,\ \sigma^2 I\right)$$

with $\sigma^2 = ln(\omega^2 + 1)$, where ω^2 is an unknown constant, then the multiplicative model is examined under the fundamental hypothesis: Ω_2^0: v has the lognormal distribution with

$$E(v) = \iota \text{ and } Cov(v, v') = \omega^2 I.$$

Thus, the consideration of hypothesis Ω_2 ensures the fact that $E(Y_i) = \delta_1 \prod_{j=2}^{p} Z_{ji}^{\delta_j}$, but in (2) it is noticed that $E(y) \neq X'\beta$ and consequently the classical theory of linear models can no longer be applied.

In the following the linear model (2) is analysed under the fundamental hypothesis:

$$\Omega: u \text{ is } N(\tau\sigma^2\iota,\ \sigma^2 I) \tag{3}$$

which can also be written in the equivalent form:

$$\Omega: y \text{ is } N(X'\beta + \tau\sigma^2\iota,\ \sigma^2 I) \tag{4}$$

where τ is any known constant and σ^2 is an unknown constant. Finally we follow the way in which the results obtained are reflected for the multiplicative model.

When in (3), in particular, $\tau = 0$ and $\tau = -\frac{1}{2}$ hypotheses Ω_1 and respectively Ω_2 are obtained.

The aim of this paper is to extend some results from the traditional theory of linear models in case that in (3) $\tau \neq 0$.

Of interest is the dependence of the results to be obtained by the values of τ.

When $\tau = -\frac{1}{2}$, 0 and 1, model (1) is called mean, median and mode unbiased, respectively [2].

3.1. Least-squares estimates and normal equations

Consider the linear model (2) under the fundamental hypothesis (4)

Definition 1. A set of functions of y (statistics):

$$\hat{\beta}_1 = \hat{\beta}_1(y), \ldots, \hat{\beta}_p = \hat{\beta}_p(y) \tag{5}$$

such that the values

$$b = (b_1, \ldots, b_p)' = (\hat{\beta}_1, \ldots, \hat{\beta}_p)'$$

minimize:

$$\mathcal{S}(y, b) = \| y - X'b \|^2 \tag{6}$$

is called a set of least-squares estimates. We denote by $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_p)' = b$ a least-squares estimate of $\beta = (\beta_1, \ldots, \beta_p)'$.

The least-squares estimates of β are determined as solutions of the linear equations system:

$$\frac{\partial \mathcal{S}(y, b)}{\partial b_\nu} = 0 \quad (\nu = 1, \ldots, p)$$

which can also be written in the form:

$$XX'b = Xy \tag{7}$$

known as the system of normal equations associated with the linear model.

In solving the system of normal equations, two cases are distinguished namely rank $X = p < n$ when the solution is unique and rank $X = r < p$ when the solution is not unique. In the first case, resulting from (7) is the least-squares estimate of β:

$$\hat{\beta} = (XX')^{-1} Xy.$$

When $\tau \neq 0$ in (4) this estimate is biased with respect to β:

$$E(\hat{\beta}) - \beta = \tau\sigma^2 (XX')^{-1} X\iota.$$

Writing $X' = (\iota \vdots X_1')$ we obtain:

$$E(\hat{\beta}) - \beta = \tau\sigma^2 \left(\begin{array}{c|c} \iota'\iota & i'X_1' \\ \hline X_1 i & X_1 X_1' \end{array} \right)^{-1} \left(\begin{array}{c} \iota'\iota \\ \hline X_1 i \end{array} \right) = \tau\sigma^2 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8}$$

It is noticed from (8) that only the estimate of β_1 is biased, the estimates of $\beta_2, \ldots, \beta_p$ being unbiased. Clearly, the estimates of the original parameters are $\tilde{\delta}_1 = e^{\hat{\beta}_1}$, $\tilde{\delta}_2 = \hat{\beta}_2, \ldots, \tilde{\delta}_p = \hat{\beta}_p$. The bias of $\tilde{\delta}_1$ is easily calculable [5].

Theorem 1. For the linear model (2) the least-squares estimates of the parameters always exist; every set of such estimates verifies system (7) and every solution of this system which depends on y is a set of least-squares estimates.

For proof see [4] or [6].

The symbol $\mathcal{S}_\Omega$ will be used for designating the minimum value of $\mathcal{S}(y, b)$, under hypothesis (4):

$$\mathcal{S}_\Omega = \mathcal{S}(y, \hat{\beta}) = \| y - X'\hat{\beta} \|^2 = \| y - \hat{\eta} \|^2 \tag{9}$$

where $\hat{\beta}$ is any least-squares estimate of β and $\hat{\eta} = X'\hat{\beta}$.

If we denote by $\eta = X'\beta$, then:

$$\eta = \sum_{j=1}^{p} \beta_j \xi_j \tag{10}$$

where the $n \times 1$ vector $\xi_j (j = 1, ..., p)$ is the j^{th} column of the matrix X'. Let r be the rank of X and V_r the r-dimensional vector space generated by the vectors $\xi_1, ..., \xi_p$. It follows from (10) that under hypothesis Ω: y is $N(\eta + \tau\sigma^2\iota, \sigma^2\iota)$, $\eta \in V_r$. As $\eta \in V_r$, according to a theorem of vector algebra [5] it results that $\hat{\eta}$ is the projection of y on V_r.

3.2. The canonical form of hypothesis Ω and the estimation of σ^2

Consider the linear model (2) under the fundamental hypothesis (4) and let r be the rank of X. The vector of observations y belongs to the vector space V_n. In the orthonormal basis $\{\rho_1, ..., \rho_n\}$ with $\rho_i = (\delta_{i1}, \delta_{i2}, ..., \delta_{in})'$ it can be written:

$$y = \sum_{i=1}^{n} y_i \rho_i$$

where δ_{ij} is the Kronecker's symbol.
Let $\alpha_1, ..., \alpha_r$ be an orthonormal basis of the vector space $V_r \subset V_n$. It is known that this basis may be completed up to an orthonormal basis in V_n by adding new sectors $\alpha_{r+1}, ..., \alpha_n$.

Denoting by $\{z_i\}$ the coordinates of y relative to basis $\{\alpha_1, ..., \alpha_r, \alpha_{r+1}, ..., \alpha_n\}$ it follows that:

$$y = \sum_{i=1}^{n} z_i \alpha_i \tag{11}$$

and, consequently:

$$z_i = \alpha_i' y \qquad (i = 1, ..., n) \tag{12}$$

as can be seen by multiplying (11) by α_i', $(i = 1, ..., n)$. Relation (12) between the coordinates $\{z_i\}$ and $\{y_i\}$ of y can be written in the form:

$$z = Py \tag{13}$$

where $z = (z_1, ..., z_n)'$ and P is the orthogonal $n \times n$ matrix which has the lines α_i'.

It results that:

$$E(z) = PE(y) = P\eta + \tau\sigma^2 P\iota. \tag{14}$$

With the notation $P\eta = \zeta$ the components of are:

$$\zeta_i = \alpha_i'\eta, \qquad (i = 1, ..., n).$$

Since $z = Py$ and P is an orthogonal matrix, it results:

$$Cov(z, z') = PCov(y, y')\, P' = \sigma^2 I.$$

For writing the canonical form, it is noticed that under hypothesis (4) $\eta \in V_r$ and consequently $\alpha_i \perp \eta$ for $i > r$, hence $\zeta_i = \alpha_i'\eta = 0$ for $i > r$.

Proposition 1. By means of the orthogonal transformation P which does not depend on the unknown parameters, the fundamental hypothesis (4) can be written in the canonical form:

$$\Omega: \begin{cases} z \text{ is } N(\zeta + \tau\sigma^2 P\iota,\ \sigma^2 I) \\ \zeta_i = 0,\ i = r+1, ..., n \end{cases}$$

where $\zeta_1, ..., \zeta_r$ and σ^2 are the unknown parameters and $r = \text{rank } X$. The calculus of the elements of matrix P is of no interest because the canonical form is not used directly in the data analysis. The canonical form helps in determining the distributions of some statistics, as it will be further seen.

In this section we shall use the canonical form for representing $\mathcal{S}_\Omega$ defined in (9) in a different way.

As $\hat{\eta}$ is the projection of y on V_r and $y = \sum_{i=1}^{n} z_i\alpha_i$ it follows that

$$\hat{\eta} = \sum_{i=1}^{r} z_i\alpha_i.$$

Consequently:

$$\mathcal{S}_\Omega = \| y - \hat{\eta} \|^2 = \left\| \sum_{i=r+1}^{n} z_i\alpha_i \right\|^2 = \sum_{i=r+1}^{n} z_i^2$$

For $i > r$, $E(z_i) = \zeta_i + \tau\sigma^2\alpha_i'\iota = \tau\sigma^2\alpha_i'\iota$ and $E(z_i^2) = \{E(z_i)\}^2 + \sigma^2 =$
$= \tau^2\sigma^4(\alpha_i'\iota)^2 + \sigma^2$.

As:

$$\mathcal{S}_\Omega = \sum_{i=r+1}^{n} z_i^2 \tag{15}$$

it results that:

$$E(\mathcal{S}_\Omega) = \sum_{i=r+1}^{n} E(z_i^2) = \tau^2\sigma^4 \sum_{i=r+1}^{n} (\alpha_i' \iota)^2 + (n-r)\,\sigma^2.$$

If we define:

$$s^2 = \frac{\mathcal{S}_\Omega}{n-r} \tag{16}$$

then:

$$E(s^2) = \frac{\tau^2\sigma^4}{n-r} \sum_{i=r+1}^{n} (\alpha_i' \iota)^2 + \sigma^2$$

and thus s^2 is a biased estimate of σ^2.

4. LINEAR FUNCTIONS OF THE UNKNOWN PARAMETERS

In the following we examine the linear functions of the parameters β of model (2) under the fundamental hypothesis (4) with $\tau \neq 0$.

The classical theory about estimable functions is no longer adequate when $\tau \neq 0$ and therefore new considerations are necessary.

The notion of function admitting a biased estimate is being introduced in this paper.

The main results obtained consist in constructing confidence regions for sets of functions admitting a biased estimate as well as the construction of tests for verifying hypotheses on these functions. By particularizing $\tau(\tau = 0)$ the results found in the classical case will be retrieved.

The new concept is formulated in the following two definitions:

Definition 2. A linear function:

$$\psi = c'\beta$$

where $c = (c_1, \ldots, c_p)'$ is a known vector is called a parametric function.

Definition 3. We say that a parametric function ψ is a function admitting a biased estimate if there exists an $n \times 1$ vector of constant coefficients a such that:

$$E(a'y) = \psi + \tau\sigma^2 a'\iota \tag{17}$$

or, in other words, if it has a biased linear estimate $a'y$.

Proposition 2. Function $\psi = c'\beta$ admits a biased estimate if and only if there is an $n \times 1$ vector a so that:

$$c' = a'X'.$$

Proof. Function $\psi = c'\beta$ admits a biased estimate if and only if there exists an $n \times 1$ vector a so that (17) takes place. But $E(a'y) = a'E(y) = a'X'\beta + \tau\sigma^2 a'\iota$ and condition $a'X'\beta = c'\beta$ resulting from (17) is identically satisfied in β if and only if $a'X' = c'$.

To prove a theorem similar to the classical Gauss-Markov theorem (see theorem 2) lemma 1 which also has a correspondent in the classical theory is necessary.

Lemma 1 If $\psi = c'\beta$ admits a biased estimate and if V_r is the space generated by the columns of X' there exists a unique biased linear estimate of ψ, say $a^{*\prime}y$ with $a^* \in V_r$. If $a'y$ is any other biased estimate of ψ then a^* is the projection of a on V_r.

Proof. As ψ admits a biased estimate, there is an $n \times 1$ vector a for which equality (17) is satisfied. It is known from vector algebra that for the vector $a \in V_n$ there are vectors $a^* \in V_r$ and $b \perp V_r$ such that $a = a^* + b$ and this decomposition is unique. In other words, a^* is the projection of a on V_r.

It results from $b \perp V_r$ that b is orthogonal to the columns of X', thus $b'X' = 0'$ and, consequently $b'X'\beta = 0$.

From (17) we obtain:

$$\begin{aligned}\psi &= E(a'y) - \tau\sigma^2 a'\iota = E(a^{*\prime}y) + E(b'y) - \\ &- \tau\sigma^2 a^{*\prime}\iota - \tau\sigma^2 b'\iota = E(a^{*\prime}y) + b'X'\beta + \\ &+ \tau\sigma^2 b'\iota - \tau\sigma^2 a^{*\prime}\iota - \tau\sigma^2 b'\iota = \\ &= E(a^{*\prime}y) - \tau\sigma^2 a^{*\prime}\iota.\end{aligned}$$

Hence:

$$E(a^{*\prime}y) = \psi + \tau\sigma^2 a^{*\prime}\iota \tag{18}$$

that is $a^{*\prime}y$ is a biased linear estimate of ψ with $a^* \in V_r$. Let us suppose that there would exist another biased linear estimate of ψ, $\alpha'y$ with $\alpha \in V_r$.

Then

$$\begin{aligned}0 = \psi - \psi &= E(a^{*\prime}y) - \tau\sigma^2 a^{*\prime}\iota - E(\alpha'y) + \tau\sigma^2\alpha'\iota = \\ &= (a^* - \alpha)' X'\beta + \tau\sigma^2(a^* - \alpha)'\iota - \tau\sigma^2(a^* - \alpha)'\iota = \\ &= (a^* - \alpha)' X'\beta\end{aligned}$$

and consequently $(a^* - \alpha)'X' = 0'$ that is $a^* - \alpha \perp V_r$ and, as $a^* - \alpha \in V_r$ it results that $a^* - \alpha = 0$ and the uniqueness of $a^{*\prime}y$ is proved.

Theorem 2. Under hypothesis (4) with $\tau \neq 0$ any function $\psi = c'\beta$ admitting a biased estimate has a unique biased linear estimate $\hat{\psi}$ with the property of a minimum variance in the class of all biased linear estimates; $\hat{\psi}$ can be obtained from $\psi = c'\beta$ by replacing β by any least-squares estimate.

To prove theorem 2 see the similar proof of the Gauss-Markov theorem in [4] or [6]. The proof is based on the fact that $a^{*\prime}y = c'\hat{\beta} = \hat{\psi}$.

Definition 4. For any function ψ admitting a biased estimate, its estimate $\hat{\psi}$ with a minimum variance whose existence and structure was given in theorem 2, is called the least-squares (biased) estimate of ψ.

Remark. The estimate $\hat{\psi}$ is a linear combination only of $z_1, ..., z_r$. Indeed, $\hat{\psi} = a^{*\prime} y = c'z$ with $c' = a^{*\prime} P'$. By denoting $c = (c_1, ..., c_n)'$ we can also write $\hat{\psi} = \sum_{j=1}^{n} c_j z_j = \sum_{j=1}^{r} c_j z_j$ because from $a^* \in V_r$ it results that $a^* \perp \alpha_j$ for for $j > r$ and, consequently $c_j = a^{*\prime} \alpha_j = 0$ for $j > r$.

5. SIMULTANEOUS ESTIMATION OF FUNCTIONS ADMITTING A BIASED ESTIMATE; TESTS FOR VERIFYING HYPOTHESIS ON THESE FUNCTIONS

Consider the linear model (2) under the fundamental hypothesis

$$\Omega: \begin{cases} y \text{ is } N(X'\beta + \tau\sigma^2 \iota, \sigma^2 I), \text{ with } \tau \neq 0 \\ \text{rank } X' = r. \end{cases} \tag{19}$$

Let $\psi_1, \psi_2, ..., \psi_q$ be any set of q functions admitting a biased estimate:

$$\psi_i = \sum_{j=1}^{p} c_{ij} \beta_j \qquad (i = 1, ..., q) \tag{20}$$

and let $\hat{\psi}_1, \hat{\psi}_2, ..., \hat{\psi}_q$ be their least-squares (biased) estimates.

From the proof of theorem 2 it results that:

$$\hat{\psi}_i = \sum_{j=1}^{n} a_{ij} y_j \tag{21}$$

with the vector $a_i = (a_{i1}, ..., a_{in})'$, $(i = 1, ..., q)$ belonging to the space V_r generated by the columns of matrix X'.

Coefficients $\{a_{ij}\}$ in (21) can be calculated in two ways, as can be seen in [5] or [7].

We can write (20) and (21) in the form:

$$\Psi = C\beta \tag{20'}$$

respectively

$$\hat{\Psi} = Ay \tag{21'}$$

where $\Psi = (\psi_1, ..., \psi_q)'$, $\hat{\Psi} = (\hat{\psi}_1, ..., \hat{\psi}_q)'$,

$$C^{(q \times p)} = (c_{ij}) \quad \text{and} \quad A^{(q \times n)} = (a_{ij}).$$

The covariance matrix of vector $\hat{\Psi}$ is:

$$Cov(\hat{\Psi}, \hat{\Psi}') = A\,Cov(y, y')\,A' = \sigma^2 AA' = \sigma^2 B \tag{22}$$

if we use the notation $AA' = B$.

A biased estimate of σ^2 in (22) is s^2 given by (16).

Theorem 3. Under the fundamental hypothesis (19) the random vector $\hat{\Psi} - \Psi$ is $N(\tau\sigma^2 A\iota, Cov(\hat{\Psi}, \hat{\Psi}'))$ and is statistically independent of $\mathcal{S}_\Omega/\sigma^2$ which has a noncentral χ^2 distribution with $n - r$ degrees of freedom.

Proof.

The fact that $\hat{\Psi} - \Psi$ has a multidimensional normal distribution results from (21') where y has this distribution. It can be easily seen that $\hat{\Psi} - \Psi$ is $N(\tau\sigma^2 A\iota, Cov(\hat{\Psi}, \hat{\Psi}'))$.

As in canonical representation $\hat{\psi}_i$ is a function only of $\{z_1, ..., z_r\}$ and $\mathcal{S}_\Omega$ only of $\{z_{r+1}, ..., z_n\}$ and because these two sets are statistically independent it results that $\hat{\Psi}$ and $\mathcal{S}_\Omega$ are statistically independent.

From (15):

$$\mathcal{S}_\Omega/\sigma^2 = \sum_{i=r+1}^{n} (z_i/\sigma)^2$$

and since $\{z_i/\sigma\}$, $(i = r + 1, ..., n)$ are independent random variables distributed $N(\iota\alpha\alpha_i'\iota, 1)$ it results that $\mathcal{S}_\Omega/\sigma^2$ is $\chi^2_{n-r,\delta}$ with

$$\delta^2 = \tau^2\sigma^2 \sum_{i=r+1}^{n} (\alpha_i'\iota)^2.$$

5.1. Confidence regions for sets of functions admitting a biased estimate

Consider the linear model (2) under the fundamental hypothesis (19) and let $\psi_1, ..., \psi_q$ be a set of q functions admitting a biased estimate. Let $\Psi = C\beta$ be the vector corresponding to these functions.

The construction of a confidence region for the point Ψ is pursued.

Suppose that functions $\psi_1, ..., \psi_q$ are linearly independent (rank $C = q$), otherwise if there are $m < q$ functions ψ_i linearly independent, the rest of the functions are linear combinations of them. By a renotation so that $\psi_{m+1}, ..., \psi_q$ are linear combinations with known constant coefficients of $\psi_1, ..., \psi_m$ which are linearly independent, it is noticed that for any position of point $(\psi_1, ..., \psi_m)$, point $(\psi_1, ..., \psi_q)$ is uniquely determined. Thus, if a confidence region for the first point is obtained, the confidence region for the second point is well determined.

It is known from theorem 3 that $\hat{\Psi} - \Psi$ is $N(\tau\sigma^2 A\iota, Cov(\hat{\Psi}, \hat{\Psi}'))$. By applying a well known theorem [7] it results that

$$(\hat{\Psi} - \Psi)' \, Cov\,(\hat{\Psi}, \hat{\Psi}')^{-1} (\hat{\Psi} - \Psi)$$

is a χ^2_{q,δ_1} random variable with the noncentrality parameter δ_1 given by:

$$\delta_1^2 = (\tau\sigma^2 A\iota)' \, Cov\,(\hat{\Psi}, \hat{\Psi}')^{-1} (\tau\sigma^2 A\iota).$$

By using (22) it results that:

$$(\hat{\Psi} - \Psi)' \frac{B^{-1}}{\sigma^2} (\hat{\Psi} - \Psi)$$

is a $\chi^2_{q_1}\delta_1$ random variable with

$$\delta_1^2 = (\tau\sigma^2 A\iota)' \frac{B^{-1}}{\sigma^2} (\tau\sigma^2 A\iota) = \tau^2\sigma^2 (A\iota)' B^{-1} A\iota \tag{23}$$

and according to theorem 3 it is independent of $\mathcal{S}_\Omega/\sigma^2$ which is $\chi^2_{n-r,\,\delta_2}$ with

$$\delta_2^2 = \tau^2\sigma^2 \sum_{i=r+1}^{n} (\alpha_i'\iota)^2. \tag{24}$$

Consequently:

$$q^{-1}(\hat{\Psi} - \Psi)' B^{-1}(\hat{\Psi} - \Psi)/s^2 \quad \text{is} \quad F^{*)}_{q,\,n-r;\,\delta_1,\,\delta_2} \tag{25}$$

with s^2 defined in (16).

Still remaining to prove is the fact that matrix B is non-singular. Because from (21'), $E(\hat{\Psi}) = AX'\beta + \tau\sigma^2 A\iota$ and from (18) $E(\hat{\Psi}) = \Psi + \tau\sigma^2 A\iota = C\beta + \tau\sigma^2 A\iota$ it results that $AX'\beta = C\beta$ identically in β, thus $C = AX'$. Then $q = \text{rank } C = \text{rank } AX' \leqslant \text{rank } A \leqslant q$ or rank $A = q$ and as $B = AA'$, it follows that rank $B = \text{rank } A = q$, that is matrix B is non-singular.

Considering (25) a confidence region for Ψ corresponding to the confidence coefficient $1 - \alpha$ is:

$$(\Psi - \hat{\Psi})' B^{-1}(\Psi - \hat{\Psi}) \leqslant q s^2 F_{\alpha;\,q,\,n-r;\,\delta_1,\,\delta_2} \tag{26}$$

where $F_{\alpha;\,q,\,n-r;\,\delta_1,\,\delta_2}$ is the upper α — quantile of the variable $F_{q,\,n-r;\,\delta_1,\,\delta_2}$ *)

Inequality (26) determines an ellipsoid in the q-dimensional space of Ψ, having its centre in point $\hat{\Psi}$. The probability that this random ellipsoid covers the point Ψ is $1-\alpha$ no matter the values of the unknown parameters $\beta_1, ..., \beta_p$, and σ^2.

In case $q = 1$, the confidence region (26) is reduced to a confidence interval for $\psi = c'\beta$.

5.2. A test of the hypothesis H: $\psi_1 = ... = \psi_q = 0$ deriving from the confidence ellipsoid

Consider the linear model (2) under the fundamental hypotheses (19) and let:

$$H: \psi_1 = ... = \psi_q = 0$$

where ψ_i, $(i = 1, ..., q)$ are q functions admitting a biased estimate and are linearly independent. These functions can be written in the form (20').

*) In practice approximations of the doubly noncentral F-variable by a central F-variable can be used.

We shall denote by ω the set of hypotheses Ω and H, that is $\omega = \Omega \cap H$ and with $\Omega - \omega$ the alternative of hypothesis H within the fundamental hypothesis Ω.

Hypothesis H is rejected at a significance level α if and only if the confidence ellipsoid (26) does not cover the point $(\psi_1, ..., \psi_q) = (0, ..., 0)$ consequently, if and only if:

$$\hat{\Psi}' B^{-1} \hat{\Psi} > q s^2 F_{\alpha; q, n-r; \delta_1, \delta_2}. \tag{27}$$

Consider $\psi_1^*, ..., \psi_q^*$ a set of q^* linearly independent functions admitting a biased estimate and consider:

$$H^*: \psi_1^* = ... \psi_q^* = 0. \tag{28}$$

Let Ψ^* be the vector corresponding to these functions:

$$\Psi^* = C^* \beta$$

with the $q^* \times p$ matrix C^* known.

Theorem 4. The necessary and sufficient condition for hypothesis H^* to be equivalent to hypothesis H is that a nonsingular $(q \times q)$ matrix D exists so that $C^* = DC$ and $q^* = q$.

For proof see [4] or [6].

Further, we shall analyse the effect on the test in case that different sets of functions admitting a biased estimate are used to define the same hypothesis H.

If the same hypothesis H is defined once as $\Psi = C\beta = 0$ and once as $\Psi^* = C^*\beta = 0$, then $C^* = DC$ and matrix D is nonsingular.

When hypothesis H is tested by using the analoque of (27) for Ψ^* the critical region is

$$\hat{\Psi}^{*\prime} B^{*-1} \hat{\Psi}^* > q s^2 F_{\alpha; q, n-r; \delta_1, \delta_2}. \tag{29}$$

The tests based on (27) and (29) are the same. Indeed, as $\hat{\Psi}^* = C^* \hat{\beta} = DC\hat{\beta} = D\hat{\Psi}$ where $\hat{\beta}$ is any vector of least-squares estimates, it results that:

$$B^* = \sigma^{-2} Cov(\hat{\Psi}^*, \hat{\Psi}^{*\prime}) = \sigma^{-2} DCov(\hat{\Psi}, \hat{\Psi}') D' = DBD'$$

and, consequently:

$$\hat{\Psi}^{*\prime} B^{*-1} \hat{\Psi}^* = \hat{\Psi}' D' (DBD')^{-1} D \hat{\Psi} = \hat{\Psi}' B^{-1} \hat{\Psi}.$$

5.3. Tests of the hypothesis H: $\psi_1 = ... = \psi_q = 0$ based on statistics $\mathcal{F}$.

It can be easily shown that hypotheses:

$$\omega: \begin{cases} y \text{ is } N(X'\beta + \tau\sigma^2 \iota, \sigma^2 I), \text{ rank } X' = r \\ H: \psi_1 = ... = \psi_q = 0 \end{cases} \tag{30}$$

and:

$$\omega' : \begin{cases} y \text{ is } N(\eta + \tau\sigma^2 \iota, \sigma^2 I), & \eta \in V_r \\ \eta \in V_{r-q}, \text{ with } V_{r-q} \end{cases} \tag{31}$$

a $(r-q)$ dimensional subspace of V_r are equivalent.

Form (30) is that used previously and (31) is useful in geometrical arguments. V_r in (31) is the space generated by the columns of X' in (30) and V_{r-q} is the subspace to which is constrained $\eta = X'\beta$ because of conditions $\psi_1 = \ldots = \psi_q = 0$.

To prove the equivalence between (30) and (31) see the similar proof in [4] or [6] with the observation that also in the present case $\Psi = A\eta$ as follows from (20') and (21') and under hypothesis ω, $\Psi = A\eta = 0$.

Hypothesis H will be tested by means of statistics:

$$\mathcal{F} = \frac{n-r}{q} \cdot \frac{\mathcal{S}_\omega - \mathcal{S}_\Omega}{\mathcal{S}_\Omega} \tag{32}$$

where $\mathcal{S}_\Omega$ is given in (9) and $\mathcal{S}_\omega$ is the minimum value of $\mathcal{S}(y, b)$ in (6) under hypotheses ω. Denoting by $\hat{\beta}\omega$ the least-squares estimate of β when hypothesis ω is taking place, we can write:

$$\mathcal{S}_\omega = \mathcal{S}(y, \hat{\beta}_\omega) = \| y - \hat{\eta}_\omega \|^2$$

with $\hat{\eta}_\omega = X'\hat{\beta}_\omega$ the projection of y on V_{r-q}.

5.3.1. *The canonical form of hypotheses ω. The distribution of $\mathcal{F}$.*

Hypothesis Ω and H determine three vector spaces: $V_{r-q} \subset V_r \subset V_n$.

Consider in V_{r-q} vectors $\{\alpha_{q+1}, \ldots, \alpha_r\}$ which form an orthonormal basis. As known, the set of these vectors may be completed with other vectors, say $\{\alpha_1, \ldots, \alpha_q\}$ so that the set $\{\alpha_1, \ldots, \alpha_q, \alpha_{q+1}, \ldots, \alpha_r\}$ should be an orthonormal basis in V_r.

This basis may be completed with vectors $\{\alpha_{r+1}, \ldots, \alpha_n\}$ so that the set $\{\alpha_1, \ldots, \alpha_q, \alpha_{q+1}, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n\}$ should form an orthonormal basis in V_n.

The canonical form of hypotheses Ω was obtained in 3.2. Since under hypothesis ω, $\eta \in V_{r-q}$ and proceeding similarly the canonical form of hypothesis H is obtained.

$$H : \zeta_i = 0, \qquad (1 \leqslant i \leqslant q) \tag{33}$$

Consequently, the canonical form of ω is:

$$\omega : \begin{cases} z \text{ is } N(\zeta + \tau\sigma^2 Pi, \sigma^2 I) \\ \zeta_i = 0 \text{ for } 1 \leqslant i \leqslant q \text{ and } r+1 \leqslant i \leqslant n. \end{cases}$$

In the following, we shall examine the distribution of $\mathcal{F}$ in (32) under hypothesis Ω and ω.

From (15) and:

$$\mathcal{S}_\omega = \| y - \hat{\eta}_\omega \|^2 = \left\| \sum_{i=1}^{n} z_i \alpha_i - \sum_{i=q+1}^{r} z_i \alpha_i \right\|^2 =$$

$$= \left\| \sum_{i=1}^{q} z_i \alpha_i + \sum_{i=r+1}^{n} z_i \alpha_i \right\|^2 = \sum_{i=1}^{q} z_i^2 + \sum_{i=r+1}^{n} z_i^2$$

it results that:

$$\mathcal{S}_\omega - \mathcal{S}_\Omega = \sum_{i=1}^{q} z_i^2. \tag{34}$$

To obtain the distribution of $\mathcal{F}$ under hypothesis Ω the following will be observed: the sets of random variables $\{z_{r+1}, ..., z_n\}$ and $\{z_1, ..., z_q\}$ in (15) and (34) are statistically independent, hence $\mathcal{S}_\Omega$ and $\mathcal{S}_\omega - \mathcal{S}_\Omega$ are statistically independent.

As $(\mathcal{S}_\omega - \mathcal{S}_\Omega)/\sigma^2 = \sum_{i=1}^{q} (z_i/\sigma)^2$ and z_i/σ are independent random variables, $N(E(z_i)/\sigma, 1)$ with $E(z_i) = z_i + \tau\sigma^2\alpha_i\iota$, it results that:

$(\mathcal{S}_\omega - \mathcal{S}_\Omega)/\sigma^2$ is a χ^2_q, δ random variable with

$$\delta^2 = \sum_{i=1}^{q} \{E(z_i)/\sigma\}^2. \tag{35}$$

From theorem 3 it is known that:

$$\mathcal{S}_\Omega/\sigma^2 \text{ is } \chi^2_{n-r,\ \delta_2}.$$

with

$$\delta_2^2 = \tau^2\sigma^2 \sum_{i=r+1}^{n} (\alpha_i'\iota)^2. \tag{36}$$

Thus, under hypothesis Ω the statistics $\mathcal{F}$ is $F_{q, n-r;\ \delta, \delta_2}$.

Under hypothesis ω:

$E(z_i) = \zeta_i + \tau\sigma^2\alpha_i'\iota = \tau\sigma^2\alpha_i'\iota$ for $\iota = 1, ..., q$, hence:

$(\mathcal{S}_\omega - \mathcal{S}_\Omega)/\sigma^2$ is a χ^2_q, δ_ω random vaiable where:

$$\delta_\omega^2 = \tau^2\sigma^2 \sum_{i=1}^{q} (\alpha_i'\iota)^2. \tag{37}$$

It results that under hypothesis ω the statistics $\mathcal{F}$ is $F_{q, n-r;\ \delta_\omega, \delta_2}$ with δ_ω^2 and δ_2^2 in (37) and (36).

Hypothesis ω is rejected at a significance level α if and only if:

$$\mathcal{F} > F_{\alpha;\ q, n-r;\ \delta_\omega, \delta_2}. \tag{38}$$

The test power is calculated according to formula:

$$\pi = P(F_{q,\,n-r;\,\delta,\,\delta_2} > F_{\alpha;\,q,\,n-r;\,\delta_{\omega},\,\delta_2})$$

with δ^2 in (35).

5.3.2. *The equivalence of the two tests of the hypothesis*

$$H\colon \psi_1 = \ldots = \psi_q = 0.$$

In 5.2. a test was obtained based on the confidence ellipsoid which consists in rejecting hypothesis H at a significance level α if and only if:

$$\hat{\Psi}' B^{-1} \hat{\Psi} > q s^2 F_{\alpha;\,q,\,n-r;\,\delta_1\,\delta_2}$$

with δ_1^2 and δ_2^2 given respectively by (23) and (24).

In section 5.3.1. the test based on statistics $\mathcal{F}$ was considered. The critical region (38) may also be written:

$$\mathcal{S}_{\omega} - \mathcal{S}_{\Omega} > q s^2 F_{\alpha;\,q,\,n-r;\,\delta_{\omega 1}\,\delta_2}.$$

To prove that these two tests are equivalent it suffice to show that:

$$\hat{\Psi}' B^{-1} \hat{\Psi} = \mathcal{S}_{\omega} - \mathcal{S}_{\Omega} \tag{39}$$

and

$$\delta_1^2 = \delta_{\omega}^2. \tag{40}$$

Firstly, we prove that equality (39) takes place.

It was pointed out in section 5.2. that $\hat{\Psi}' B^{-1} \hat{\Psi}$ has the same value no matter the set of q linearly independent functions admitting a biased estimate used in describing hypothesis H. Consequently, functions $\psi_i^* = \zeta_i$, $(i = 1, \ldots, q)$ can be chosen for describing hypothesis H. This choice is motivated in the first place by (33).

Functions ψ_i^* are also linearly independent and admit a biased estimate.

The fact that they are linear functions of $\{\beta_j\}$ results from $\zeta = Pn = PX'\beta$. They are linearly independent because, if for $\{c_1, \ldots, c_q\}$ not all zero it results $\sum_{i=1}^{q} c_i \zeta_i \equiv 0$, identically in $\{\beta_j\}$, then it follows that for $\zeta_1, \ldots, \zeta_q$ not all values are allowed.

Functions $\{\psi_i^*\}$ admit a biased estimate; $\hat{\psi}_i^* = z_i$ is a biased estimate of ζ_i, as results from $E(Z_i) = \zeta_i + \sigma\tau^2\alpha'_i\iota$. As $z_i = \alpha'_i y$ is a biased estimate of $\zeta_i = \alpha'_i \eta$ and for $i = 1, \ldots, q$, $\alpha_i \in V_r$ it follows that $\psi_i^* = Z_i$ is just the least-squares estimate of $\psi_i^* = \zeta_i$.

It follows that:

$$\hat{\Psi}' B^{-1} \hat{\Psi} = \hat{\Psi}^{*\prime} B^{*-1} \hat{\Psi}^{*} = \hat{\Psi}^{*\prime} \hat{\Psi}^{*} = \sum_{i=1}^{q} \hat{\psi}_i^{*2} = \sum_{i=1}^{q} z_i^2$$

because

$$B^{*} = \sigma^{-2} Cov(\hat{\Psi}^{*}, \hat{\Psi}^{*\prime}) \text{ and } Cov(\hat{\Psi}^{*}, \hat{\Psi}^{*\prime}) = \sigma^2 I^{(q \times q)}. \tag{41}$$

Thus:

$$\hat{\Psi}' B^{-1} \hat{\Psi} = \sum_{i=1}^{q} z_i^2$$

From (34) and (41) follows (39). It should be still pointed out that $\delta_1^2 = \delta_\omega^2$. Because $E(\hat{\Psi}^{*} - \Psi^{*}) = \tau\sigma^2 P_1 \iota$ where P_1 is the matrix having as elements the first q lines of matrix P and writing δ_1^2 in (23) in the equivalent form:

$$\delta_1^2 = (\tau\sigma^2 P_1 \iota)' \frac{I}{\sigma^2} (\tau\sigma^2 P_1 \iota)$$

(40) is resulting.

5.3.3. *Calculus of the noncentrality parameters* δ *and* δ_2.

In (35) the noncentrality parameter δ is expressed by means of the canonical form parameters.

In the following, we shall obtain the expression of δ^2 in terms of the parameters of the original problem.

Taking into account the fact that $z_i = \alpha_i' y$ and $E(z_i) = \zeta_i + \tau\sigma^2 \alpha_i' \iota$ and by replacing these formulas in $\mathcal{S}_\omega - \mathcal{S}_\Omega = \sum_{i=1}^{q} z_i^2$, respectively $\sigma^2\delta^2 = \sum_{i=1}^{q} \{E(z_i)\}^2$ we obtain:

$$\mathcal{S}_\omega - \mathcal{S}_\Omega = \sum_{i=1}^{q} (\alpha_i' y)^2 \tag{42}$$

and

$$\sigma^2\delta^2 = \sum_{i=1}^{q} (\zeta_i + \tau\sigma^2 \alpha_i' \iota)^2. \tag{43}$$

The following rule for the calculus of δ^2 is inferred from (42) and (43):

Rule 1. Under hypothesis Ω the value of the noncentrality parameter δ of the distribution of statistics $\mathcal{F}$ is calculated as follows:

At the numerator of $\mathcal{F}$, in $\mathcal{S}_\omega - \mathcal{S}_\Omega$ we replace the vector of observations y by its mean and the result is $\sigma^2\delta^2$.

We similarly determine δ_2^2 in (36), namely from $\mathcal{S}_\Omega = \sum_{i=r+1}^{n} z_i^2$ we obtain:

$\mathcal{S}_\Omega = \sum_{i=r+1}^{n} (\alpha_i' y)^2$ wherefrom:

Rule 2. The value of the noncentrality parameter δ_2 is calculated as follows:

At the denominator of $\mathcal{F}$ in $\mathcal{S}_\Omega$ we replace the vector of observations y by its mean and the result in $\sigma^2\delta_2^2$. The effective calculus of the noncentrality parameters is made by using some practical considerations [7].

In conclusion, the results obtained for the linear model are easily transposed in case of the multiplicative model, if the notations made in (2) are taken into account. By writting $\psi = c'\beta$ as $\psi = c_1 \ln \delta_1 + c_2 + \ldots + c_p\delta_p$ the results for functions admitting a biased estimate can be transposed in the case of parametric functions e^ψ of interest for the initial model $Y_i = e^{\ln\delta_1 + x_{2i}\delta_2 + \ldots + x_{pi}\delta_p + u_i}$ $(i = 1,\ldots,n)$.

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ANALYSE STATISTIQUE DES MODELES MULTIPLICATIFS DANS LE CAS D'UNE TRANSFORMATION LOGARITHMIQUE

L'auteur observe une classe de modèles multiplicatifs pouvant être linéarisés par transformation logarithmique. Pour étudier les fonctions paramétriques on introduit la notion de fonction qui admet l'estimateur à biais et on établit un théorème du type Gauss-Markov. On a construit, utilisant les propriétés des estimateurs des paramètres et des fonctions susceptibles à être estimées, des régions de confiance pour ces dernièrés.

L'auteur résout un probleme important de l'économétrie, étant donné que les modèles multiplicatifs ont pour cas particulier la classe des fonctions de production du type Cobb-Douglas.

СТАТИСТИЧЕСКИЙ АНАЛИЗ МУЛЬТИПЛИКАЦИОНЫХ МЕТОДОВ В СЛУЧАЕ ЛОГАРИФМИЧЕСКИХ ПРЕВРАЩЕНИЙ

В работе ведётся изучение класса мультипликационных моделей, которые могут быть линиаризированы логарифмическими превращениями. Для изучения параметрических функций вводится понятие функции, позволяющей перенесённое исчисление и устанавливается теорема типа Гаусса-Маркова. На основе свойств исчислений параметров и функции, подлежащие исчислению, построены районы доверия для этих функцій.

Работа решает важный вопрос экономометрии, так как мультипликационные модели содержат частным образом класс производственных функций типа Кобба-Дугласа.

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