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On Fuzzy Environment in Optimization Problems

The paper deals with problems related to fuzziness in optimizing processes. The fuzziness is supposed to be modelled by fuzzy sets.

The main problem is that of linear programming with fuzzy sets coefficients defined as a natural extension of linear programming with inclusion coefficients.

Further on, there are presented two problems related to multicriteria optimization bearing in mind a shading on criteria set and a selection problem of a Pareto efficient point according to criteria defined by fuzzy sets.

1. INTRODUCTION

The purpose of this paper is to treat decision problems in which the objectives and restraining conditions are not required to be precisely formulated. We want to learn to identify such problems, characterize the properties of their solutions, and, to the extent possible, master techniques for solving them.

There has recently been a growing awareness that a deterministic model may not be the best to represent the real environment. [1].

By a decision situation we understand: a set of alternatives which we assume to be at the disposal of the free choice, a set of constraints on the choice and an objective function which allows the decision-maker to rank the alternatives.

A decision structure under certainty has the property that the adoption of a particular choice is known to lead invariably to one specific value of the objective function. This means that there exists no special difficulty in the way of explaining preferances vis-à-vis outcomes, i.e., that each decision-maker is in posession of a set of clearly formulated goals on the basis of which he is able to determine his own preferences.

If we now consider some real decision situations, we may well feel that they do not satisfy the above definition.

It is patent that the real decision-maker uses expressions as "z should be in the vecinity of $\tilde{z}$" which is not a clearly formulated goal.

The expression "... in the vecinity of ..." can be modelled by a fuzzy subset, that is a function $A_i : X \to L$, where L is a lattice.

A fuzzy environment can be viewed, thus, as a set X of alternatives together with his fuzzy subsets, representing ambigously formulated criteria (constraints and goals), i.e. a system $(X, A_1, \ldots A_n, L)$.

To do the best taking into account all the criteria, as well as possible, means to build a function $D = A_1 \cap A_2 \cap \ldots \cap A_n$. Optimum means that part of X which maximises D [2].

Bearing in mind the fact that constraints and goals can be treated identically in the formulation of a decision, we are ready to extend the confluence principle and define a fuzzy environment as the mathematical object (X, D, L). This line of thought suggests that a decision can be defined as a fuzzy subset of alternatives.

By optimal decision we simply mean a point $x_0 \in X$ (if any) with $D(x_0) = \sup_{x \in X} D(x)$

It is perhaps fair to stress at this point, that any fuzzy programming problem is a multicriteria optimization problem, and that the correspondence $(X, A_1, A_2, \ldots, A_n, L) \to x_0$ is a compact notation of the general programming problem previously described as associating to every family of goals and constraints a subset of the alternatives set, called the optimal set.

The business of this paper is to develop a mathematical framework within which we can not only talk about fuzzy environment, but can also derive important results which help us to solve programming problems in such environment.

In many cases, even though the deterministic decision structure fit the real situation the decision-maker is not able to systematically evaluate the consequences which would follow from data imprecision.

The very process of attempting to formalize decision problems in this manner forces us to consider tolerances on parameters [3], [4]. A tolerance, viewed as an interval of variation can be described by a fuzzy set.

This is precisely the cause of linear programming with fuzzy constraints.

In many real problems comparison between alternatives must be made on the basis of multiple heterogeneous and complex consequences. Then, the unique objective function is replaced by many. Much more, we can rank this objective functions, according to an external preference. This is precisely the case of weighted confluence in fuzzy programming.

Finally both tolerances and weighted confluence are considered.

2. PREREQUISITES

Let X be a set and L the unit interval. A fuzzy subset of X is a function $A : X \to L$.

We say that X is the carrier of A, L the truth value set and $A(x)$ is the membership degree of x at A.

If A and B are fuzzy subsets of X, their intersection denoted by $A \cap B$ is given by

$$(A \cap B)(x) = A(x) \wedge B(x)$$

and their union, denoted by $A \cup B$ is given by

$$(A \cup B)(x) = A(x) \vee B(x)$$

for each x is X.

Denoting by $\mathfrak{F}(X)$ the set $\{A \mid A : X \to L\}$ we observe that the structure of L induces the same structure on $\mathfrak{F}(X)$. If $A, B \in \mathfrak{F}(X)$, we say that A is included in B if $A(x) \leqslant B(x)$ for each x in X.

If $A \in \mathfrak{F}(X)$, $r \in L$, the r — weak level of A is the set $\omega_r(A) = \{x \in X \mid A(x) \geqslant r\}$, and the r — strong level is the set $\sigma_r(A) = \{x \in X \mid A(x) > r\}$.

From the preceding definitions it is readily shown that the level sets posess the following properties:

(1) $\omega_0(A) = X$	$\sigma_0(A) = \operatorname{supp} A, \quad \sigma_1(A) = \varnothing$
(2) $r_1 \leqslant r_2 \Rightarrow \omega_{r_2}(A) \subseteq \omega_{r_1}(A)$	$r_1 \leqslant r_2 \Rightarrow \sigma_{r_2}(A) \subseteq \sigma_{r_1}(A)$
(3) $\omega_r(A \cup B) = \omega_r(A) \cup \omega_r(B)$	$\sigma_r(\bigcup_i A_i) = \bigcup_i \sigma_r(A_i)$
(4) $\omega_r(\bigcap_i A_i) = \bigcap \omega_r(A_i)$	$\sigma_r(A \cap B) = \sigma_r(A) \cap \sigma_r(B)$
(5) $A \subseteq B \Leftrightarrow \omega_r(A) \subseteq \omega_r(B), \ \forall r$	$A \subseteq B \Leftrightarrow \sigma_r(A) \subseteq \sigma_r(B), \ \forall r$

2.1. A representation theorem for fuzzy subsets

Let be $A \in \mathfrak{F}(X)$, $r < 1$. Then, since $x \in \sigma_r(A) \Rightarrow A(x) > r$, there is $r' > r$ such that $A(x) > r'$ and $x \in \sigma_{r'}(A)$

Let us now consider a mapping

$$f : [0, 1] \to \mathfrak{P}(X)$$

where $\mathfrak{P}(X)$ is the power set of X, with the following properties

a) $f(1) = \varnothing$

b) $r_1 \leqslant r_2 \Rightarrow f(r_2) \subseteq f(r_1)$

c) $x \in f(r) \Rightarrow \exists r' > r$ with $x \in f(r')$

We are ready now to evaluate the strong levels of A by the following:

Theorem [5]: There is a bijection $\mathfrak{T}$ between the set $\{f \mid f : [0, 1] \to \mathfrak{P}(X)\}$ and the set $\mathfrak{F}(X)$ such that $f(r)$ is a strong level.

Proof: We shall define $\mathfrak{T}(f) : X \to L$ by

$$\mathfrak{T}(f)(x) = \begin{cases} \sup \{r \mid x \in f(r)\} & \text{for} \quad x \in f(0) \\ 0 & \text{otherwise} \end{cases}$$

We have to show that $f(r) = \sigma_r(\mathcal{F}(f))$. Let be $x \in \sigma_r(\mathcal{F}(f))$. Then $\mathcal{F}(f)(x) > r$, and, thus, there is $r' > r$ such that $x \in f(r')$.

But $f(r') \subseteq f(r)$ and, then, $x \in f(r)$, i.e., $\sigma_r(\mathcal{F}(f)) \subseteq f(r)$ Conversely, let be $x \in f(r)$. Then, bearing in mind the property (b), there is $r' > r$ with $x \in f(r')$. Therefore, $\mathcal{F}(f)(x) \geqslant r' > r$ and imediately it follows $f(r) \subseteq \sigma_r(\mathcal{F}(f))$.

In order to prove the injectivity, suppose $\mathcal{F}(f_1) = \mathcal{F}(f_2)$ that is, $\sigma_r(\mathcal{F}(f_1)) = \sigma_r(\mathcal{F}(f_2))$ or $f_1(r) = f_2(r)$
This means $f_1 = f_2$.

2.2. Fuzzy subsets on vector spaces

The carrier X, as defined in the previous section, is an entity without internal mathematical structure. The set in which we are interested, such as point sets on a line or plane, can be given structures.

If we have placed a vector space structure on the carrier of fuzzy subsets, we might well wish to know wheather we can define a fuzzy subset on this carrier. The problem is to see if the operations of addition and multiplication can be extended to $\mathcal{F}(E)$, where E is a vector space.

Before doing so, however, we will need some further background on properties of fuzzy sets.

If $f : X \to Y$ is a mapping and $A \in \mathcal{F}(X)$ then the image of A through f is the fuzzy subset $f(A) : Y \to L$ given by:

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

If $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, then $A \times B \in \mathcal{F}(X \times Y)$ is the product of fuzzy subsets defined by

$$(A \times B)(x, y) = A(x) \wedge B(y)$$

We are ready now to define addition and multiplication in $\mathcal{F}(E)$ as follows:

1) if $A, B \in \mathcal{F}(E)$ the sum of the fuzzy subsets A and B is a fuzzy subset of E given by the image of the cartesian product $A \times B$ through addition,

2) if $A \in \mathcal{F}(E)$ then the product which associates with each scalar $\lambda \in \boldsymbol{R}$ and A a unique element in $\mathcal{F}(E)$ denoted by λA, is given by the image of A through the assignment $x \mapsto \lambda x$

Definition 1: A fuzzy vector subspace of E is a fuzzy subset $A \in \mathcal{F}(E)$ with the properties

i) $A(x_1 + x_2) \geqslant A(x_1) + A(x_2)$, $x_1, x_2 \in E$

ii) $A(\lambda x) \geqslant A(x)$, $\lambda \in \boldsymbol{R}$, $x \in E$

Proposition 1: $A \in \mathcal{F}(E)$ is a fuzzy vector subspace of E, if and only, if, the levels $\sigma_r(A)$ are vector subspace of E, for all $r \in [0, 1]$.

Proof: Suppose that A is a fuzzy vector subspace, we have $\sigma_1(A) = \varnothing$ and by definition we take $\varnothing$ as a vector subspace of E. From condition (ii) it results $A(0) \geqslant A(x)$, $x \in E$, that is $A(0) = \sup\limits_{x \in E} A(x)$.

Thus $\forall r \geqslant A(0)$, $\sigma_r(A) = \varnothing$. Clearly, for each $r < A(0)$, $\sigma_r(A)$ is a vector space.
Definition 2: A fuzzy subset $A \in \mathcal{F}(E)$ is said to be convex if for each $x_1, x_2 \in X$, $\lambda \in [0, 1]$, $A[\lambda x_1 + (1 - \lambda)x_2] \geqslant A(x_1) \wedge A(x_2)$
Example: Any fuzzy vector subspace of E is convex, since

$$A[\lambda x_1 + (1 - \lambda)x_2] \geqslant A[\lambda x_1] \wedge A[(1 - \lambda)x_2] \geqslant A(x_1) \wedge A(x_2)$$

Proposition 2: $A \in \mathcal{F}(E)$ is convex if and only if $\sigma_r(A)$ are convex for all $r \in [0, 1]$
Proof: obvious

Proposition 3: Any intersection of fuzzy vector subspace (convex fuzzy subsets) is a fuzzy vector subspace (convex subset)
Proof: obvious

Proposition 4: Let be $A, B \in \mathcal{F}(E)$ and $\lambda \in \boldsymbol{R}$. Then

i) $\sigma_r(A + B) = \sigma_r(A) + \sigma_r(B)$

ii) $\sigma_r(\lambda A) = \lambda \sigma_r(A)$, if $\bigvee_{x \in E} A(x) = 1$

Proof: We have defined addition of fuzzy subsets by

$$(A + B)(z) = \bigvee_{(x,\, y) \in E^2} [A(x) \wedge B(y)], \quad x + y = z, \quad z \in E$$

If $z \in \sigma_r(A + B)$, then $\bigvee_{\substack{x,\, y \in E^2 \\ x+y=z}} [A(x) \wedge B(y)] > r$ and there exists x, y, $A(x)$, $B(y) > r$, $x \in \sigma_r(A)$ and $y \in \sigma_r(B)$ $x + y = z$; therefore $z \in \sigma_r(A) + \sigma_r(B)$. Consequently, we have proved that $\sigma_r(A + B) \subseteq \sigma_r(A) + \sigma_r(B)$. Since $\sigma_r(A) + \sigma_r(B) \subseteq \sigma_r(A + B)$ we have finally $\sigma_r(A + B) = \sigma_r(A) + \sigma_r(B)$.

We have defined multiplication by

$$(\lambda A)(z) = \begin{cases} A(u_\lambda), & u_y = \lambda^{-1} z \quad \text{for } \lambda \neq 0 \\ 0 \quad \text{for } \lambda = 0, & z \neq 0 \\ \bigvee_{x \in E} A(x) \quad \text{for } \lambda = 0, & z = 0 \end{cases}$$

Let be $z \in \sigma_r(\lambda A)$. Then $A(u_\lambda) > r$ i.e. $z = \lambda u_\lambda \in \lambda \sigma_r(A)$. Therefore $\sigma_r(\lambda A) \subseteq \lambda \sigma_r(A)$. Similarly, we can show that $\lambda \sigma_r(A) \subseteq \sigma_r(\lambda A)$.
In this way we have proved that on strong level addition and multiplication satisfies the representation theorem.
Proposition 5: Let $A, B \in \mathcal{F}(E)$ be convex fuzzy subsets and then $A + B$ and λA are convex fuzzy subsets.

3. LINEAR PROGRAMMING WITH FUZZY CONSTRAINTS

Our understanding of what we shall mean by a basic single-stage model in connection with decision making under fuzziness will be governed by the definitions which follow:

Let the mapping $f : \boldsymbol{R}^n \to \boldsymbol{R}$ be a linear function $f(x) = \langle c, x \rangle$, $c \in \boldsymbol{R}^n$ and a feasible region

$$P = \{x = (x_1, x_2, ..., x_n) \in \boldsymbol{R}_+^n \mid x_1 A_1 + x_2 A_2 + ... x_n A_n \subseteq A\}$$

where A_j are fuzzy subsets $A_j : \boldsymbol{R}^m \to [0, 1]$, A is a fuzzy subset $A : \boldsymbol{R}^m \to [0, 1]$ and the binary operation $+$ refers to addition of fuzzy sets as defined in section 2.2.

The problem is then to find $x^\circ \in P$ such that:

$$\langle c, x^\circ \rangle = \inf_{x \in P} \langle c, x \rangle$$

Bearing in mind proposition 4, we are concerned with the following:

Proposition 6: $x = (x_1, ..., x_n) \in \boldsymbol{R}_+^n \cap P$ if and only if $\forall r \in [0, 1]$

$$\text{(a)} \quad \begin{cases} x_1 \sigma_r(A_1) + x_2 \sigma_r(A_2) + ... + x_n \sigma_r(A_n) \subseteq \sigma_r(A) \\ x_j \geqslant 0, \; j = 1, ..., n \end{cases}$$

The feasible region is specifed via set containtment by an infinite number of constraints as defined by Soyster [6].

Proposition 7: If the fuzzy subset $A : \boldsymbol{R}^m \to [0, 1]$ has the property:

$$A(\boldsymbol{R}^m) = \{r_1, r_2, ..., r_p\}$$

$$0 = r_1 < r_2 < ... < r_p \leqslant 1$$

then $x = (x_1, x_2, ..., x_n) \in \boldsymbol{R}_+^n \cap P$ if and only if

$$\text{(a')} \quad \begin{cases} x_1 \sigma_{r_i}(A_1) + x_2 \sigma_{r_i}(A_2) + ... + x_n \sigma_{r_i}(A_n) \subset \sigma_{r_i}(A) \\ x_j \geqslant 0, \; j = 1, \; ..., \; n, \; i = 1, \; ..., \; p \end{cases}$$

Proof: obvious if $x \in \boldsymbol{R}_+^n \cap P$. Conversely, suppose (a') and let be $r \in [0, 1]$. Then, for each $i \in \{1, 2, ..., p\}$ we have $r_i \leqslant r \leqslant r_{i+1}$.

For $r = r_i$ or $r = r_{i+1}$, (a) is verifed. If $r_i < r < r_{i+1}$ then $\sigma_r(A) = \sigma_{r_i}(A)$. Clearly, $\sigma_r(A) \subseteq \sigma_{r_i}(A)$. If we suppose $x \in \sigma_{r_i}(A)$ such that $x \notin \sigma_r(A)$, than, $r_i < A(x) \leqslant r < r_{i+1}$ which contradicts the hypothesis.

Therefore,

$$x_1 \sigma_r(A_1) + ... + x_n \sigma_r(A_n) \subseteq x_1 \sigma_{r_i}(A_1) + ... + x_n \sigma_{r_i}(A_n) \subseteq \sigma_{r_i}(A) = \sigma_r(A)$$

On the basis on the proposition 2, the fuzzy programming problem is appropiately classified as an inexact programming problem with finite constraints. This simplification permits the optimal solution to be obtained with an auxiliary linear programming problem as set forth by Soyster [6] if:

(1) $A_i : \boldsymbol{R}^m \to [0, 1]$, $i = 1, ..., n$ are convex

(2) $A(\boldsymbol{R}^m) = \{r_1, r_2, ..., r_p\}$, $\quad 0 = r_1 < r_2 < ... < r_p \leqslant 1$

(3) $\sigma_{r_i}(A) = \{y \in \boldsymbol{R}^m \mid y \leqslant b^{r_i}\}$, $i = 1, ..., p$, $b^{r_i} \in \boldsymbol{R}^m$.

According to the previous propositions, the inexact data in the columns $i = 1, ..., p$ of linear programms can be modelled by defining $A_i : \boldsymbol{R}^m \to [0, 1]$.

Taking a sampling of the unit interval [0, 1] and defining $A : \boldsymbol{R}^m \to [0, 1]$ with the properties (2), (3), the problem (a') with set-inclusive constraints can be easly solved.
To illustrate this fuzzy extension of linear programming consider the following example which satisfies to (1), (2), (3).
The sampling of the unit interval is

$$0 < \frac{1}{6} < \frac{1}{2} < \frac{4}{5} < 1$$

the fuzzy feasible region is

$$xA_1 + yA_2 \subset A$$

$$A_1 : \boldsymbol{R}^2 \to [0, 1] \text{ with } A_1(x, y) = f_1(x) \wedge g_1(y)$$

$$A_2 : \boldsymbol{R}^2 \to [0, 1] \text{ with } A_2(x, y) = f_2(x) \wedge g_2(y)$$

$$f_1, f_2, g_1, g_2 : \boldsymbol{R} \to [0, 1]$$

$$f_1(x) = \begin{cases} 2 - x & \text{for } x \in [1, 2] \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} 4 - x & \text{for } x \in [3, 4] \\ 0 & \text{otherwise} \end{cases}$$

$$g_1(y) = \begin{cases} \frac{y-1}{2} & \text{for } y \in [1, 3] \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(y) = \begin{cases} y - 4 & \text{for } y \in [4, 5] \\ 0 & \text{otherwise} \end{cases}$$

$$A : \boldsymbol{R}^2 \to [0, 1]$$

$$A(x, y) = \begin{cases} 1 & \text{for } (x, y) \leqslant (4, 1) \\ 4/5 & \text{for } (x, y) \leqslant (6, 3) \text{ and } (x, y) \not\leqslant (4, 1) \\ 1/2 & \text{for } (x, y) \leqslant (8, 4) \text{ and } (x, y) \not\leqslant (6, 3) \\ 1/6 & \text{for } (x, y) \leqslant (10, 5) \text{ and } (x, y) \not\leqslant (8, 4) \\ 0 & \text{otherwise} \end{cases}$$

Clearly, (1), (2), (3) are verifed. For instance at the level 1/2, the constraint will be:

$$x\sigma_{1/2}(A_1) + y\sigma_{1/2}(A_2) \subseteq \sigma_{1/2}(A), \qquad x \geqslant 0, \; y \geqslant 0$$

where

$$\sigma_{1/2}(A_1) = \{(u, v) \in \boldsymbol{R}^2 \mid u \in [1, 3/2), v \in (1, 3]\} = [1, 3/2) \times (1, 3]$$

$$\sigma_{1/2}(A_2) = [3, 7/2) \times (9/2, 5]$$

$$\sigma_{1/2}(A) = \{w \in \boldsymbol{R}^3 \mid w \leqslant [6, 3)\}$$

4. WEIGHTED CONVOLUTION IN FUZZY PROGRAMMING

Our understanding of what we mean by the notion of a fuzzy environment in conection with decision-making is governed by the following definition: A fuzzy environment is a complex $(X, A_1, ..., A_n, L)$ where X is the alternative set, L is a lattice, and $A_i : X \to L$, $i = 1, ..., n$ are fuzzy criteria.

Much of the decision-making in the real world takes place in a fuzzy environment in which the goals must be combined when they are of unequal importance.

In this case, to the notion of fuzzy environment we simply need to add the notion of L-weighting in order to be in position to render a clear restatement of the basic decision problem.

Denoting by N_n the set of criteria indices $\{1, ..., n\}$ an L-weighting is viewed as a mapping $g : \mathfrak{P}(N_n) \to L$. This mapping must be a fuzzy measure [7], which models a preference, that is the importance degree of the different coalitions formed with the elements of N_n, in fact with the n fuzzy sets of a fuzzy environment.

For each $x \in X$ we consider a mapping

$$h_x : N_n \to L$$

given by $h_x(i) = A_i(x)$

Let be $D : X \to L$ a decision function defined as a fuzzy integral [8]:

$$D(x) = \int_{N_n} h_x(i) \circ g = \sup_{M \in \mathfrak{P}(N_n)} \inf_{i \in M} [h_x(i) \wedge g(M)]$$

An element $\tilde{x} \in X$ maximizing $D(x)$ is an optimal decision.

In order to illustrate these ideas in a concrete setting it may be well to consider at this point a method to determine $\tilde{x} \in X$ such that $D(\tilde{x}) = \sup_{x \in X} D(x)$

We shall denote by $\mathfrak{P}(n)$ the set of all permutations of N_n.

Proposition 8:

$$\sup_{x \in X} D(x) = \max_{\pi \in \mathfrak{P}^*(n)} \{ \max_{k=1, ..., n} [\sup_{x \in X(\pi)} (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)})]\}$$

where

$$\mathfrak{P}^*(n) = \{\Pi \in \mathfrak{P}(n) \mid \exists x \in X \text{ with } A_{\pi(i)}(x) \leqslant A_{\pi(i+1)}(x),\ 1 \leqslant i \leqslant n-1\}$$

$$X(\Pi) = \{x \in X \mid A_{\pi(i)}(x) \leqslant A_{\pi(i+1)}(x),\ 1 \leqslant i \leqslant n-1\}$$

$$\alpha_{\pi(k)} = g(\{\Pi(k),\ \Pi(k+1),\ ...,\ \Pi(n)\}),\ k = 1,\ ...,\ n$$

Proof: we shall use the following lemma due to Sugeno [7].

Let be $K = \{s_1, s_2, ..., s_n\}$, $g : \mathfrak{P}(K) \to [0, 1]$ a fuzzy measure on K, and $h : K \to [0, 1]$ a function.

If $h(s_i) \leqslant h(s_{i+1})$, $1 \leqslant i \leqslant n-1$, then

$$\int_K h(s) \circ g = \max_i [h(s_i) \wedge g(K_i)]$$

where $K_i = \{s_i, s_{i+1}, ..., s_n\}$, $i = 1, ..., n$

On the basis of this lemma, we have

$$D(x) = \max_k [A_{\pi(k)}(x) \wedge \alpha_{\pi(k)}]$$

where $\Pi \in \mathfrak{P}(n)$ which for $x \in X(\Pi)$
Let be $\Pi \in \mathfrak{P}^*(n)$, and $x \in X(\Pi)$. Then

$$\alpha_0 = \sup_{x \in X} D(x) \geqslant D(x)$$

that is

$$\sup_{x \in X} D(x) \geqslant \sup_{x \in X(\pi)} [\max_k (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)})] =$$

$$= \max_k [\sup_{x \in X(\pi)} (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)})]$$

Since $\Pi \in \mathfrak{P}^*(n)$ is arbitrarly, it results

$$\alpha_0 \geqslant \max_{\pi \in \mathfrak{P}^*(n)} \{\max_k [\sup (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)})]\} = \alpha_1$$

Conversely, let be $x \in X$. Then there is $\Pi \in \mathfrak{P}^*(n)$ such that $x \in X(\Pi)$ and

$$\alpha_1 \geqslant \max_k [\sup_{x \in X(\pi)} (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)}] =$$

$$= \sup_{x \in X(\pi)} [\max_k (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)})] \geqslant \max_k (A_{\pi(k)}(x) \wedge \alpha_{\pi(k)}) = D(x)$$

Since x is arbitrarly, it results $\alpha_1 \geqslant \sup_x D(x) = \alpha_0$ and proposition 1 is proved.

For each $\Pi \in \mathfrak{P}^*(n)$ and $k \in Nn$, we denote by $(x(\Pi(k)), \lambda(\Pi(k)) \in X \times [0, 1]$ the optimal solution of the following programming problem:

$$P(\Pi, k) = \begin{cases} \lambda = \max! \\ \lambda \leqslant A_{\pi(k)}(x) \\ \lambda \leqslant \alpha_{\pi(k)} \\ x \in X(\Pi) \end{cases}$$

Corollary: An alternative $\tilde{x} \in X$ is an optimal decision if and only if $\tilde{x} = \tilde{x}(\Pi_0(k_0))$ for $\Pi_0 \in \mathfrak{P}^*(n)$ and $k_0 \in Nn$ with the property

$$\lambda(\Pi_0(k_0)) = \max_{\pi \in \mathfrak{P}^*(n)} \max_k k \ \lambda(\Pi(k))$$

Proof: By proposition 8, noting that

$$\lambda(\Pi(k)) = A_{\pi(k)}(x^*) \wedge \alpha_{\pi(k)} = \sup_{x \in X(\pi)} [A_{\pi(k)}(x) \wedge \alpha_{\pi(k)}]$$

is equivalent with the fact that $(x^*, \lambda(\Pi(k))$ is the optimal solution of problem $P(\Pi, k)$.

5. LINEAR PROGRAMMING WITH FUZZY OBJECTIVE FUNCTIONS

Whereas, in section 3, we concentrated upon multiple constraints, we now pay attention to the equally important case of multiple goals.

Let be the following optimization problem: find $\max f(x)$ subject to

$$\Omega \begin{cases} Ax \leqslant b \\ x > 0 \end{cases}$$

where

$$f(x) = [f_1(x), f_2(x), \ldots, f_p(x)], \; f_i(x) = \sum_{j=1}^{n} c_{ij} \, x_j$$

$$A = (a_{ij}), \; i = 1, \ldots, m, \; j = 1, \ldots, n, \; b \in \boldsymbol{R}^m, \; x \in \boldsymbol{R}^n$$

and $\max f(x)$ means an efficient (Pareto) point $x^* \in \boldsymbol{R}^n$ of f on Ω. Suppose given a vector $a = (a_1, a_2, \ldots, a_p) \in \boldsymbol{R}^p$. We want an x^* Pareto with the property $f_i(x^*) \geqslant a_i$, $i = 1, \ldots, p$. The numbers a_i must be thought as minimal values accepted for the criteria $f_i(x)$.

This problem can be framed into the fuzzy programming problem with fuzzy constraints [9] (see also [8]).

Let us consider that Ω is compact. Let $f^* = (f_1^*, f_2^*, \ldots, f_p^*) \in \boldsymbol{R}^p$ be the ideal point $f_i^* = \sup_{x \in \Omega} f_i(x)$ and suppose $f_i^* > a_i$, $i = 1, \ldots, p$.

For every i, we define the following fuzzy subset of Ω

$$g_i : \Omega \to [0, 1]$$

$$g_i(x) = \begin{cases} \dfrac{f_i(x) - a_i}{f_i - a_i} & \text{if } f_i(x) \geqslant a_i \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \Omega$, $g_i(x)$ represents the membership degree of the value $f_i(x)$ at the interval $[a_i, f_i^*]$.

The optimal decision $x^* \in \Omega$ satisfies.

$$D(x^*) = \sup_{x \in \Omega} D(x)$$

where $D : \Omega \to [0, 1]$

$$D(x) = \min_i g_i(x)$$

This problem is equivalent with the following linear programming problem [4]:

$$(P) \begin{cases} x_{n+1} = \max ! \\ Ax \leqslant b \\ f_i(x) - (f_i^* - a_i) x_{n+1} \geqslant a_i \end{cases}$$

If $\Omega \neq \varnothing$, this problem has a solution

Theorem: Suppose $\exists x^\circ \in \Omega$ such that $f_i(x^\circ) \geqslant a_i$ and let (x^*, x^*_{n+1}) be a solution of the problem (P).

Then,

(a) $x^* \in \boldsymbol{R}^n$ is an efficient (Pareto) point of f on Ω and $f_i(x^*) \in [a_i, f_i^*]$

(b) Let be $A_i = a_i + (f_i^* - a_i)x^*_{n+1}$

There is no $(B_1, B_2, ..., B_p) \in \boldsymbol{R}^p$ with $B_i > A_i$ such that the system

$$Ax \leqslant b$$

$$f_i(x) \geqslant B_i, \quad i = 1, ..., p$$

$$x \geqslant 0$$

is compatible.

Proof: a) It is sufficient to show that x^* is an efficient (Pareto) point for $g = (g_1, g_2, ..., g_p)$. We have

$$\min_i g_i(x^*) = \sup_{x \in \Omega} \min_i g_i(x)$$

Let be $x \in \Omega$ such that $g(x) \leqslant g(x^*)$ or $g(x) \geqslant g(x^*)$. Then $\min_i g_i(x) \leqslant$ $\leqslant \min_i g_i(x^*) \leqslant g_i(x^*)$.

Let be i_0 with $g_{i_0}(x) = \min_i g_i(x)$. Then $g_{i_0}(x) < g_{i_0}(x^*)$ that is $g(x) \leqslant$ $\leqslant g(x^*)$ and x^* is an efficient (Pareto) point for g. By hypothesis $(x^\circ, 0)$ is a solution of the problem (P), and, therefore, $x^*_{n+1} > 0$, i.e.,

$$f_i(x^*) \in [a_i, f_i^*], \quad i = 1, ..., p$$

b) Suppose that there is $(B_1, B_2, ..., B_p) \in \boldsymbol{R}^p$ with $B_i > A_i$, $i =$ $= 1, ..., p$ satisfing the compatibility.

Then $\exists x^\circ \in \Omega$ such that $f_i(x^\circ) \geqslant B_i$ and (x°, x°_{n+1}) with $x^\circ_{n+1} =$ $= \min_i \dfrac{f_i(x^\circ) - a_i}{f_i^* - a_i}$ verifies the constraints of the problem (P).

We have also $f_i(x^\circ) > A_i = a_i + (f_i^* - a_i)x^*_{n+1}$, i.e.

$$x^*_{n+1} < \frac{f_i(x^\circ) - a_i}{f_i^* - a_i} \quad \text{or} \quad x^*_{n+1} < x^\circ_{n+1}$$

which contradicts the statement that (x^*, x^*_{n+1}) is an optimal solution of (P).

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DU MILIEU FUZZY DANS LES PROBLÈMES D'OPTIMISATION

L'auteur traite de quelques aspects concernant le saisissement de l'imprécision dans les problèmes d'optimisation. On suppose que l'imprécision est mise en modèle par des ensembles fuzzy.

L'un des problèmes est celui de la programmation linéaire à coefficients ensembles fuzzy définie en tant qu'une extension naturelle de la programmation linéaire ayant des restrictions du type inclusion.

On traite aussi de deux problèmes de l'optimisation ayant plusieurs critères tenant compte des nuances dans l'ensemble des critères et un problèmes de sélection d'un point de maximum Pareto selon les critères définis par des ensembles fuzzy.

О СРЕДЕ ФАЗИ В ВОПРОСЕ ОПТИМИЗАЦИИ

В работе трактуются некоторые аспекты, связанные с выделением неточности в вопросе оптимизации. Предполагается, что неточность была моделирована при помощи множестаа фази.

Проблемой является линейное программирование с коэффициентами множества фази, определяемое как естественное расширение прямолинейного программирования с ограничениями типа включения.

Далее тракутются две проблемы, связанные с многокритериальной оптимизацией, учитывающей нюансы на множество критериев и проблему отбора максимальной точки Пасета по критериям, определенным путем использования множества фази.