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Initial Distributions for Limit Laws of Extreme Values

In a previous paper [1] we have shown how to set the problem and how limit probabilistic laws for extreme values are to be found. A question remained open, namely that of the initial distributions F for which the maximum M_n, respectively the minimum m_n, of a sample having a volume n tends towards one of the three limit laws which were found when $n \to \infty$. Particularly, we will deal with the initial distributions which lead to limit distributions of the Weibull type, in view to gain a better insight as regards the extent which one can use the Weibull distribution for the statistical problem of concern.

We will start with a particular case of the Feller lemma [2], or better said Hincin -Feller lemma, because Hincin is the one who discovered the asymptotic property of distributions pertaining to this type (see for exemple [3] page 39), and Feller gave it the final form which we used in [1] and will use also here because it spares the use of other additional lemmas (e.g. lemmas 1 and 2, section 3 [4], the first lemma being also presented in [3] page 41).

In the frame of this paper we will perfect several demonstrations to be found in literature concerning certain distributions F, some results will be extended and others, relatively recent, will be presented briefly.

1. Suppose that there exist two pairs of sequences of real numbers $\{a_n\}$; $\{b_n\}$; $\{\alpha_n\}$; $\{\beta_n\}$; the first and the third sequence being composed of positive numbers, so that in every point of continuity of the distribution function U: $\lim_{n\to\infty} F_n(a_n x + b_n) = U(x)$; $\lim_{n\to\infty} F_n(\alpha_n x + \beta_n) = U(x)$. We ask ourselves which is the relation between the numbers a_n, b_n, and α_n, β_n.

We are in the frame of the Hincin-Feller lemma. Instead of $V(x) = U(Ax + B)$ we have $V(x) = U(x)$, thus $A = 1$, $B = 0$. According to the above-mentioned lemma:

$$\lim_{n\to\infty} \frac{\alpha_n}{a_n} = 1; \qquad \lim_{n\to\infty} \frac{\beta_n - b_n}{a_n} = 0$$

and, reciprocally, if these conditions are met and one of the sequences $F_n(a_n x + b_n)$, $F_n(\alpha_n x + \beta_n)$ tends towards $U(x)$, and this in every continuity point of U, then also the other sequence tends towards the same limit in the respective point.

Let us proceed now to the maximum problem. Here $F_n(x) = F^n(x)$. If we suppose that $F^n(a_n x + b_n) \to U(x)$ in every point of continuity of $U(x)$ with $0 < U(x) < 1$, then necessarely $F(a_n x + b_n) \to 1$ when $n \to \infty$, because if there was a c such that $0 < c < 1$, so that an infinity of numbers n for which $F(a_n x + b_n) < c$ exists, then the limit of $F^n(a_n x + b_n)$ would be 0, or wouldn't exist, which is contrary to the hypothesis. But as

$$F^n(a_n x + b_n) = \left([1 - (1 - F(a_n x + b_n))]^{\frac{1}{1-F(a_n x+b_n)}}\right)^{n[1-F(a_n x+b_n)]}$$

it follows that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = e^{-\lim\limits_{n\to\infty} n[1-F(a_n x+b_n)]}$$

and as

$$\lim_{n\to\infty} F^n(a_n x + b_n) = U(x) = e^{\ln U(x)}$$

it is clear that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = U(x) \Leftrightarrow \lim_{n\to\infty} n[1 - F(a_n x + b_n)] = -\ln U(x);$$
$$0 < U(x) < 1.$$

The problem amounts to finding those distribution functions F and those sequences of positive real numbers, respectively real numbers $\{a_n\}$, $\{b_n\}$ such that:

$$\lim_{n\to\infty} n[1 - F(a_n x + b_n)] = -\ln U(x)$$

in every point x of U for which $0 < U(x) < 1$, U being one of the three possible distribution types.

The above-stated condition may be written also as follows:

$$n[1 - F(a_n x + b_n)] = -\ln U(x) + \varepsilon(x, n, a_n, b_n) \quad \text{i.e.} \quad F(a_n x + b_n) =$$

$$= 1 + \frac{\ln U(x) - \varepsilon(x, n, a_n, b_n)}{n} \text{ with } \varepsilon(x, n, a_n, b_n) \to 0 \text{ when } n \to \infty.$$

The last expression, the fact that for $0 < U(x) < 1$, U is a continuous, strictly increasing function, as well as certain results obtained by Gnedenko (theorem 6 [3]) have directed us towards a relatively easy method for computing the elements of the corresponding series, $\{a_n\}$, $\{b_n\}$ namely:

Let x_1, x_2 be any two numbers $x_1 < x_2$ such that $0 < U(x_1)$, $U(x_2) < 1$. We will denote by $x_i^{(n)} (i = 1, 2)$ the smallest x for which:

$$F(x) \leqslant 1 + \frac{\ln U(x_i)}{n} \leqslant F(x+0).$$

(We recall that $F(x-0) = F(x)$ by definition, and if x is a point of continuity, than also $F(x+0) = F(x)$, as well as the fact that there exist continuous functions constant on certain intervals).

We intend to prove that one may take as a_n, b_n the solutions of the following equations:

$$a_n x_1 + b_n = x_1^{(n)}; \qquad a_n x_2 + b_n = x_2^{(n)}.$$

To this end we start from the assumption that there exist two sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$\lim_{n\to\infty} n[1 - F(\alpha_n x + \beta_n)] = -\ln U(x) \qquad \text{i.e.}$$

$$F(\alpha_n x + \beta_n) = 1 + \frac{\ln U(x) - \varepsilon(x, n, \alpha_n, \beta_n)}{n} \text{ with } \varepsilon(x, n, \alpha_n, \beta_n) \to 0$$

when $n \to \infty$.

Let some h with $0 < U(x_i - h)$, $U(x_i + h) < 1$, for $i = 1, 2$.

Then:

$$F\left[\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n\right] = 1 + \frac{U\left(x_i - \frac{h}{2}\right) - \varepsilon\left(x_i - \frac{h}{2}, n, \alpha_n, \beta_n\right)}{n};$$

$$F\left[\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n\right] = 1 + \frac{U\left(x_i + \frac{h}{2}\right) - \varepsilon\left(x_i + \frac{h}{2}, n, \alpha_n, \beta_n\right)}{n}.$$

We want that simultaneously: $F\left[\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n\right] < 1 +$

$$+ \frac{\ln U(x_i)}{n}; \text{ and } F\left[\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n\right] > 1 + \frac{\ln U(x_i)}{n}; \quad i = 1,2.$$

For this it is sufficient that:

$$\left|\varepsilon\left(x_i - \frac{h}{2}\right), n, \alpha_n, \beta_n\right| < \frac{1}{2}\left[\ln U(x_i) - \ln U\left(x_i - \frac{h}{2}\right)\right] \text{ and}$$

$$\left|\varepsilon\left(x_i + \frac{h}{2}\right), n, \alpha_n, \beta_n\right| < \frac{1}{2}\left[\ln U\left(x_i + \frac{h}{2}\right) - \ln U(x)\right]; \; i = 1, 2.$$

As each of the ε tends towards o when $n \to \infty$, there exists a finite number $N(h)$ such that for $n > N(h)$ these inequalities are satisfied. But then:

$$\alpha_n(x_i - h) + \beta_n < x_i^{(n)} = a_n x_i + b_n < \alpha_n(x_i + h) + \beta_n; \quad i = 1, 2$$

because $\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n$ can not be larger than $x_i^{(n)}$ as in this case it would be necessary that

$$F\left[\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n\right] \geqslant F(x_i^{(n)} + 0) \geqslant 1 + \frac{\ln U(x_i)}{n},$$

which is impossible since

$$F\left[\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n\right] < 1 + \frac{\ln U(x_i)}{n}.$$

Therefore $\alpha_n\left(x_i - \frac{h}{2}\right) + \beta_n \leqslant x_i^{(n)}$ and consequently:

$$\alpha_n(x_i - h) + \beta_n < x_i^{(n)}.$$

Equally $\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n$ can not be smaller than $x_i^{(n)}$ because then $F\left[\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n\right] \leqslant F(x_i^{(n)}) \leqslant 1 + \frac{\ln U(x_i)}{n}$ which is impossible since

$$F\left[\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n\right] > 1 + \frac{\ln U(x_i)}{n}.$$

Hence $\alpha_n\left(x_i + \frac{h}{2}\right) + \beta_n \geqslant x_i^{(n)}$ (more precisely $>$) and thus $\alpha_n(x_i + h) + \beta_n > x_i^{(n)}$. The double inequalities which were developed may be also written as follows:

$$x_i - h < \frac{a_n}{\alpha_n} x_i + \frac{b_n - \beta_n}{\alpha_n} < x_i + h; \quad i = 1, 2; \quad n > N(h)$$

As h is any, it follows that:

$$\lim_{n\to\infty}\left[\frac{a_n}{\alpha_n} x_1 + \frac{b_n - \beta_n}{\alpha_n}\right] = x_1, \quad \lim_{n\to\infty}\left[\frac{a_n}{\alpha_n} x_2 + \frac{b_n - \beta_n}{\alpha_n}\right] = x_2.$$

If both x_1 and x_2 are different from zero, after multiplication of the first relation through x_2, of the second through x_1, and subtracting, it becames immediately obvious that $\lim_{n\to\infty} \frac{b_n - \beta_n}{\alpha_n} = 0$. Further, subtracting this relation from the first one, it can be seen that $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 1$. Finally, if either x_1 or x_2 is equal to 0, for instance $x_1 = 0$, the first relation reduces to $\lim_{n\to\infty} \frac{b_n - \beta_n}{\alpha_n} = 0$, and subtracting if from the second, one can see that also $\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 1$.

Concluding, if $F^n(\alpha_n x + \beta_n) \to U(x)$ when $n \to \infty$, then $\frac{a_n}{\alpha_n} \to 1$; $\frac{b_n - \beta_n}{\alpha_n} \to 0$ which, based on the particular case of the Hincin-Feller lemma mentioned at the beginning, leads to $F^n(a_n x + b_n) \to U(x)$ and conversly, if $F^n(a_n x + b_n) \to U(x)$ then also $F^n(\alpha_n x + \beta_n) \to U(x)$.

Let us consider now a case often met in practice, namely that in which F is a continuous function. Here $F(x) = F(x + 0)$ and $x_i^{(n)}$ i.e. $a_n x_i + b_n$ is the smallest root of the equation $F(a_n x_i + b_n) = 1 + \frac{\ln U(x_i)}{n}$; $i = 1, 2$. Of course if F is strictly increasing the equation has an unique solution.

We will apply the obtained results to the canonical representation of the three types of limit functions possible for the maximum, namely $U(x) = e^{-e^{-x}}$; $e^{-x^{-\beta}}$; $e^{-(-x)^{\beta}}$.

a) $U(x) = e^{-e^{-x}}$; $-\infty < x < +\infty$; $\ln U(x) = -e^{-x}$ and the two equations concerning x_1 and x_2 are:

$$F(a_n x_1 + b_n) = 1 - \frac{e^{-x_1}}{n}; \quad F(a_n x_2 + b_n) = 1 - \frac{e^{-x_2}}{n}$$

It is natural to take $x_1 = 0$ and $x_2 = 1$. We arrive thus to the following conditions developed by Gnedenko, formulated as in [5] (with minute changes as regards notations).

For the existence of a limit distribution for the maximum (M_n) — corresponding to the initial distribution F — having the value $e^{-e^{-x}}$ it is necessary and sufficient that for any x; $-\infty < x < +\infty$

$$\lim_{n \to \infty} n[1 - F(a_n x + b_n)] = e^{-x}$$

where a_n and b_n are given by the equations: $F(b_n) = 1 - \frac{1}{n}$, $F(a_n + b_n) = 1 - \frac{1}{e \cdot n}$.

b) $U(x) = e^{-x^{-\beta}}$; $0 < x < +\infty$; $U(0) = 0$; $\ln U(x) = -x^{-\beta}$ and the two equations concerning x_1 and x_2 are:

$$F(a_n x_1 + b_n) = 1 - \frac{x_1^{-\beta}}{n}; \quad F(a_n x_2 + b_n) = 1 - \frac{x_2^{-\beta}}{n}.$$

Here we could take $x_1 = 1$; $x_2 = 2$. We arrive thus to the following conditions:

So that the limit distribution of the maximum (M_n), corresponding to the initial distribution F should exist and be $e^{-x^{-\beta}}$ it is necessary and sufficient that for any x, $0 < x < +\infty$:

$$\lim_{n \to \infty} n[1 - F(a_n x + b_n)] = x^{-\beta}$$

where a_n and b_n are given by the equations: $F(a_n + b_n) = 1 - \frac{1}{n}$;

$$F(2a_n + b_n) = 1 - \frac{1}{2^\beta \cdot n}$$

c) $U(x) = e^{-(-x)^\beta}$; $-\infty < x < 0$; $U(0) = 1$; $\ln U(x) = -(-x)^\beta$ and the two equations for x_1 and x_2 are:

$$F(a_n x_1 + b_n) = 1 - \frac{(-x_1)^\beta}{n}; \quad F(a_n x_2 + b_n) = 1 - \frac{(-x_2)^\beta}{n}.$$

Here one could take $x_1 = -1$; $x_2 = -\frac{1}{2}$. We arrive thus to the following conditions:

So that the limit distribution of the maximum (M_n), corresponding to the initial distribution F, should exist and be $e^{-(-x)^\beta}$ it is necessary and sufficient that for any x; $-\infty < x < 0$:

$$\lim_{n \to \infty} n[1 - F(a_n x + b_n)] = (-x)^\beta$$

where a_n and b_n are given by the equations: $F(-a_n + b_n) = 1 - \frac{1}{n}$;

$$F\left(-\frac{a_n}{2} + b_n\right) = 1 - \frac{1}{2^\beta \cdot n}$$

Of course, in the problems with practical character in which we are interested, the limit distributions appear only as approximations of the exact ones and hence for estimating the obtained approximation degree it is necessary to find a_n and b_n. Therefore the above-mentioned method for finding a_n and b_n (not only in the case of continuous initial distributions) is of great interest.

Arrises also the question of finding directly (without proceeding through a_n and b_n) the necessary and sufficient conditions which an initial distribution function has to meet so that the limit distribution of the maximum M_n should exist and pertain to one of the three possible canonical types:

$$e^{-x^{-\beta}}, \quad e^{-(-x)^\beta}, \quad e^{-e^{-x}}.$$

This problem was entirely solved by Gnedenko [4] as regards the limit distributions $e^{-x^{-\beta}}$ and $e^{-(-x)^\beta}$, but the results obtained by the former for $e^{-e^{-x}}$ are only partial. Also the following formally complete solutions, e.g. that of Marcurs and Pinsky [5], can not be considered satisfactory.

2. We will deal now with these questions. But first, a note. Those who study Gnedenko's paper, impressed by the elegance and soberness of the way the problems are treated, are tempted not to follow with sufficient attention all the details of the demonstration and thus to oversee certain shortcomings. As we will show, such is the case of the forced conclusions drawn from the realtions on row 5 page 441. In the same way, the elegance and simplicity of

certain of Feller's demonstrations [2] make us to accept some of his statements which should be proved, as in the case of the last part of the demonstration regarding the proposition page 270—271.

This is why our presentation, in view to be complete and self sufficient, will differ from certain demonstrations considered till now as satisfactory.

We will start with an observation about the Hincin-Feller lemma [2] page 246—247. If one follows attentively its demonstration, one can see immediately that it holds not only for distribution sequences $\{F_n\}$ where n pertains to the set of natural numbers, but also for $\{F_{n_k}\}$ where n_k pertains to any infinite subset of the set of natural numbers, if in the enunciation and in the demonstration n is replaced everywhere by n_k.

Let us consider some $\lambda > 1$. As $\{n_k\}$ we will consider the sequence $\{[\lambda^k]\}$ where $[\lambda^k]$ is the integer part of the number λ^k. Of course, starting from a certain k (which depends upon λ) all the numbers $[\lambda^k]$ will be different.

Let us suppose now that $F^n(\alpha_n x + \beta_n) \to e^{-x^{-\beta}}$ and consequently that

$$\text{also } F^{[\lambda^n]}[\alpha_{[\lambda^n]}\ x + \beta_{[\lambda^n]}] \to e^{-x^{-\beta}}.$$

(Any infinite subsequence of a convergent sequence is also convergent and has the same limit).

We observe now that

$$F^{[\lambda^n]}[\alpha_{[\lambda^{n+1}]}x + \beta_{[\lambda^{n+1}]}] = [F^{[\lambda^{n+1}]}(\alpha_{[\lambda^{n+1}]}x + \beta_{[\lambda^{n+1}]}]^{\frac{[\lambda^n]}{[\lambda^{n+1}]}} \to (e^{-x^{-\beta}})^{1/\lambda} = e^{-(\lambda^{1/\beta}x)^{-\beta}}$$

because $\dfrac{[\lambda^n]}{[\lambda^{n+1}]} = \dfrac{\lambda^n - \varepsilon_n}{\lambda^{n+1} - \varepsilon_{n+1}} \to \dfrac{1}{\lambda}$ when $n \to \infty$, since $0 \leqslant \varepsilon_n,\ \varepsilon_{n+1} < 1$.

Applying the Hincin-Feller lemma, completed as was shown before, it follows that

$$\frac{\alpha_{[\lambda^{n+1}]}}{\alpha_{[\lambda^n]}} \to \lambda^{1/\beta}; \qquad \frac{\beta_{[\lambda^{n+1}]} - \beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 0 \qquad \text{when } n \to \infty.$$

$$\text{As } \lambda > 1, \text{ also } \lambda^{1/\beta} > 1.$$

From the above-mentioned relations it follows in the first place that $\alpha_{[\lambda^n]} \to \infty$ when $n \to \infty$. Indeed, let $\varepsilon > 0$, be so that $\lambda^{1/\beta} - \varepsilon > 1$. From the first relation follows that there exists a natural number n_0 such that for $p \geqslant n_0$: $\lambda^{1/\beta} - \varepsilon < \dfrac{\alpha_{[\lambda^{p+1}]}}{\alpha_{[\lambda^p]}} < \lambda^{1/\beta} + \varepsilon$ and particularly $\dfrac{\alpha_{[\lambda^{p+1}]}}{\alpha_{[\lambda^p]}} > \lambda^{1/\beta} - \varepsilon$ for $p = n_0, n_0 + 1, \ldots, n - 1$.

Multiplying the corresponding inequalities it follows that $\alpha_{[\lambda^n]} > \alpha_{[\lambda^{n_0}]} \cdot (\lambda^{1/\beta} - \varepsilon)^{n-n_0}$ and hence $\alpha_{[\lambda^n]} \to \infty$ when $n \to \infty$.

In the second place we will prove that $\dfrac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 0$ when $n \to \infty$. Since $\dfrac{\beta_{[\lambda^{p+1}]} - \beta_{[\lambda^p]}}{\alpha_{[\lambda^p]}} \to 0$ when $p \to \infty$, it follows that to every $\varepsilon > 0$ corresponds

a natural number $n_0 = n_0(\varepsilon)$ so that for $p \geqslant n_0$:

$$-\frac{\lambda^{1/\beta}-1}{4}\varepsilon < \frac{\beta_{[\lambda^{p+1}]}-\beta_{[\lambda^p]}}{\alpha_{[\lambda^p]}} < \frac{\lambda^{1/\beta}-1}{4}\varepsilon\ ;\quad -\frac{\lambda^{1/\beta}-1}{4}\varepsilon\,\alpha_{[\lambda^p]} <$$

$$< \beta_{[\lambda^{p+1}]} - \beta_{[\lambda^p[} < \frac{\lambda^{1/\beta}-1}{4}\varepsilon\,\alpha_{[\lambda^p]}$$

Making $p = n_0, n_0+1, \ldots, n-1$ and adding up, we obtain

$$-\frac{\lambda^{1/\beta}-1}{4}\varepsilon[\alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}] < \beta_{[\lambda^n]} - \beta_{[\lambda^{n_0}]} <$$

$$< \frac{\lambda^{1/\beta}-1}{4}\varepsilon[\alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}]$$

and consequently:

$$-\frac{\lambda^{1/\beta}-1}{4}\varepsilon\frac{\alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}}{\alpha_{[\lambda^n]}} + \frac{\beta_{[\lambda^{n_0}]}}{\alpha_{[\lambda^n]}} < \frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} <$$

$$< \frac{\lambda^{1/\beta}-1}{4}\varepsilon\frac{\alpha_{[\lambda^{n_0}} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}}{\alpha_{[\lambda^n]}} + \frac{\beta_{[\lambda^{n_0}]}}{\alpha_{[\lambda^n]}}.$$

As n_0 is fixed and $\alpha_{[\lambda^n]} \to \infty$ when $n \to \infty$, there exists a natural number $n_1 =$ $= n_1(\varepsilon)$ such that for $n > n_1$: $-\frac{\varepsilon}{2} < \frac{\beta_{[\lambda^{n_0}]}}{\alpha_{[\lambda^n]}} < \frac{\varepsilon}{2}$.

Further we will recall a well known theorem of O. Stolz: Let us consider the sequences $\{U_n\}$; $\{V_n\}$. If starting from a certain n:

a) $V_{n+1} > V_n$; b) $\lim_{n\to\infty} V_n = +\infty$; c) if $\lim_{n\to\infty}\frac{U_{n+1}-U_n}{V_{n+1}-V_n}$ exists, then

also $\lim_{n\to\infty}\frac{U_n}{V_n}$ exists and $\lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{U_{n+1}-U_n}{V_{n+1}-V_n}$.

We will now apply this theorem taking $U_n = \alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} +$ $+ \ldots + \alpha_{[\lambda^{n-1}]}$; $V_n = \alpha_{[\lambda^n]}$.

Of course, the choosen V_n meets the requirements a) and b) of the theorem; then, as $U_{n+1} - U_n = \alpha_{[\lambda^n]}$; $V_{n+1} - V_n = \alpha_{[\lambda^{n+1}]} - \alpha_{[\lambda^n]}$ it is evident that

$$\lim_{n\to\infty}\frac{U_{n+1}-U_n}{V_{n+1}-V_n} = \lim_{n\to\infty}\frac{\alpha_{[\lambda^n]}}{\alpha_{[\lambda^{n+1}]}-\alpha_{[\lambda^n]}} = \frac{1}{\lim\limits_{n\to\infty}\frac{\alpha_{[\lambda^{n+1}]}}{\alpha_{[\lambda^n]}} - 1} = \frac{1}{\lambda^{1/\beta}-1}$$

Consequently condition c) is met as well, so that:

$$\frac{\alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}}{\alpha_{[\lambda^n]}} \to \frac{1}{\lambda^{1/\beta} - 1}$$

Thus, starting from a certain n, say n_2:

$$\frac{\alpha_{[\lambda^{n_0}]} + \alpha_{[\lambda^{n_0+1}]} + \ldots + \alpha_{[\lambda^{n-1}]}}{\alpha_{[\lambda^n]}} < \frac{2}{\lambda^{1/\beta} - 1}$$

From the double inequality relating to $\frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}}$ it follows by far that for $n > max\ [n_0, n_1, n_2]$:

$$-\frac{\lambda^{1/\beta} - 1}{4} \varepsilon \frac{2}{\lambda^{1/\beta} - 1} - \frac{\varepsilon}{2} < \frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} < \frac{\lambda^{1/\beta} - 1}{4} \varepsilon \frac{2}{\lambda^{1/\beta} - 1} + \frac{\varepsilon}{2} \quad ; \quad -\varepsilon < \frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} < \varepsilon.$$

Thus, to every $\varepsilon > 0$ corresponds a $N(\varepsilon)$ such that for $n > N(\varepsilon)$:

$$\left| \frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \right| < \varepsilon, \quad \text{i.e. } \lim_{n \to \infty} \frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} = 0.$$

Hence, if $F^n(\alpha_n x + \beta_n) \to e^{-x^{-\beta}}$ then of course also

$$F^{[\lambda^n]}(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) \to e^{-x^{-\beta}}$$

and according to the above-stated $\frac{\beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 0$. If we take now $a_{[\lambda^n]} = \alpha_{[\lambda^n]}$; $b_{[\lambda^n]} = 0$ one can see that $\frac{a_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 1$; $\frac{\beta_{[\lambda^n]} - b_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 0$ and consequently, by virtue of the completed Hincin-Feller lemma:

$$F^{[\lambda^n]}(a_{[\lambda^n]} x + b_{[\lambda^n]}) \to e^{-x^{-\beta}} \qquad \text{i.e.} \quad F^{[\lambda^n]}(\alpha_{[\lambda^n]} x) \to e^{-x^{-\beta}}.$$

We will use this last result to show how the necessary and sufficient condition of Gnedenko can be obtained, namely the condition F must meet so that $M_n \to e^{-x^{-\beta}}$.

3. We have seen that $F^n(\alpha_n x + \beta_n) \to e^{-x^{-\beta}} \Rightarrow F^{[\lambda^n]}(\alpha_{[\lambda^n]} x) \to e^{-x^{-\beta}}$ and that $F^n(\alpha_n x + \beta_n) \to e^{-x^{-\beta}} \Leftrightarrow n(1 - F(\alpha_n x + \beta_n)) \to x^{-\beta}$.

Exactly in the same manner as was demonstrated previously for the last relation, it can be shown that:

$$F^{[\lambda^n]}(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) \to e^{-x^{-\beta}} \Leftrightarrow [\lambda^n](1 - F(\alpha_{[\lambda^n]} x)) \to x^{-\beta}.$$

We recall further that from $\frac{\alpha_{[\lambda^{n+1}]}}{\alpha_{[\lambda^n]}} \to \lambda^{1/\beta}$ it followed that from a certain n, the sequence $\{\alpha_{[\lambda^n]}\}$ is increasing and $\alpha_{[\lambda^n]} \to \infty$ when $n \to \infty$.

Consequently, to every sufficiently large x there will correspond a natural number $s(x) = s$ such that

$$\alpha_{[\lambda^s]} \leqslant x < \alpha_{[\lambda^{s+1}]} \text{ and therefore } 1 - F(\alpha_{[\lambda^{s+1}]}) \leqslant 1 - F(x) \leqslant 1 - F(\alpha_{[\lambda^s]}).$$

Let us consider now some positive number K. Obviously $K\alpha_{[\lambda^s]} \leqslant \leqslant Kx \leqslant K\alpha_{[\lambda^{s+1}]}$ and therefore:

$$1 - F(\alpha_{[\lambda^{s+1}]}\, K) \leqslant 1 - F(Kx) \leqslant 1 - F(\alpha_{[\lambda^s]}\, K)$$

Hence:

$$\frac{1 - F(\alpha_{[\lambda^{s+1}]})}{1 - F(\alpha_{[\lambda^s]}\, K)} \leqslant \frac{1 - F(x)}{1 - F(Kx)} \leqslant \frac{1 - F(\alpha_{[\lambda^s]})}{1 - F(\alpha_{[\lambda^{s+1}]} K)}$$

$$\frac{[\lambda^s]}{[\lambda^{s+1}]} \cdot \frac{[\lambda^{s+1}] \cdot (1 - F(\alpha_{[\lambda^{s+1}]}))}{[\lambda^s] \cdot (1 - F(\alpha_{[\lambda^s]} K)} - K^\beta \leqslant \frac{1 - F(x)}{1 - F(Kx)}$$

$$- K^\beta \leqslant \frac{[\lambda^{s+1}]}{[\lambda^s]} \cdot \frac{[\lambda^s] \cdot (1 - F(\alpha_{[\lambda^s]}))}{[\lambda^{s+1}] \cdot (1 - F(\alpha_{[\lambda^{s+1}]} K))} - K^\beta.$$

If $x \to \infty$ than also $s \to \infty$; $\dfrac{[\lambda^{s+1}]}{[\lambda^s]} \to \lambda$; $[\lambda^s](1 - F(\alpha_{[\lambda^s]})) \to 1$; $[\lambda^s](1 - F(\alpha_{[\lambda^s]}\, K)) \to K^{-\beta}$ etc. so that the left-hand side of the above double inequality tends towards $K^\beta\left(\dfrac{1}{\lambda} - 1\right)$, and the right-hand side towards $K^\beta(\lambda - 1)$.

Let us consider some $\varepsilon > 0$. Let us take $\lambda = 1 + \dfrac{\varepsilon}{2K^\beta}$. The two limits become $-\dfrac{\varepsilon}{2} \cdot \dfrac{1}{1 + \dfrac{\varepsilon}{2K^\beta}}$ and $\dfrac{\varepsilon}{2}$. Beyond a sufficiently large value for x, say x_0, the left-hand side of the double inequality will be greater than $-\varepsilon$ $\left(\text{the limit being } -\dfrac{\varepsilon}{2} \cdot \dfrac{1}{1 + \dfrac{2}{2K^\beta}}\right)$, and the right-hand side smaller than ε $\left(\text{the limit being } \dfrac{\varepsilon}{2}\right)$. Thus, for $x > x_0(\varepsilon, K)$; $\left|\dfrac{1 - F(x)}{1 - F(Kx)} - K^\beta\right| < \varepsilon$ namely $\lim\limits_{n \to \infty} \dfrac{1 - F(x)}{1 - F(Kx)} = K^\beta$. This is Gnedenko's condition. We have shown that it is necessary. We shall prove now that it is also sufficient.

Let us suppose therefore that $\lim\limits_{n \to \infty} \dfrac{1 - F(x)}{1 - F(Kx)} = K^\beta$ for any $K > 0$. We intend to prove that there exist positive numbers a_n such that $\lim\limits_{n \to \infty} n(1 - F(a_n x)) = x^{-\beta}$.

First a note. There can not exist a finite x_1 such that $F(x_1) = 1$, because in this case the limit wouldn't exist (if x_1 was the smallest number such that

$F(x_1) = 1$, and taking for x larger values than $max(x_1, Kx_1)$, the expression $\frac{1 - F(x)}{1 - F(Kx)}$ wouldn't make sense and we couldn't speak of $\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(Kx)}$. Consequently, for every finite x, $F(x) < 1$, and of course $\lim_{x \to \infty} F(x) = 1$, F being a distribution function.

We will take for a_n the smallest x which satisfies the double inequality:

$$F(x) \leqslant 1 - \frac{1}{n} \leqslant F(x + 0) \text{ and thus } F(a_n) \leqslant 1 - \frac{1}{n} \leqslant F(a_n + 0) \leqslant F(a_n(1 + \varepsilon'))$$

with certain $\varepsilon' > 0$ (Obviously, $a_n \to \infty$ when $n \to \infty$).

Consequently:

$$n(1 - F(a_n(1 + \varepsilon')) \leqslant 1 \leqslant n(1 - F(a_n))$$

$$\text{and thus } \frac{n(1 - F(a_n x))}{n(1 - F(a_n))} \leqslant \frac{n(1 - F(a_n x))}{1} \leqslant \frac{n(1 - F(a_n x))}{n(1 - F(a_n(1 + \varepsilon')))},$$

and further:

$$\frac{1 - F(a_n x)}{1 - F(a_n)} - x^{-\beta} \leqslant n(1 - F(a_n x)) - x^{-\beta} \leqslant \frac{1 - F(a_n x)}{1 - F(a_n(1 + \varepsilon'))} - x^{-\beta}.$$

If we make $n \to \infty$, the left-hand side would tend towards 0, because in this case the condition stated in the enunciation would apply to the fraction, with $a_n x$ instead of x, and a_n instead of Kx (here x is fixed) and thus

$$\lim_{n \to \infty} \left[\frac{1 - F(a_n x)}{1 - F(a_n)} - x^{-\beta} \right] = \lim_{a_n x \to \infty} \left[\frac{1 - F(a_n x)}{1 - F(a_n)} - x^{-\beta} \right] = \left(\frac{1}{x} \right)^{\beta} - x^{-\beta} = 0.$$

Likewise, when $n \to \infty$ the right-hand side tends towards $\frac{(1 + \varepsilon')^{\beta} - 1}{x^{\beta}}$ because here, instead of x we have $a_n x$, and instead of Kx; $a_n(1 + \varepsilon')$, so that:

$$\lim_{n \to \infty} \left[\frac{1 - F(a_n x)}{1 - F(a_n(1 + \varepsilon'))} - x^{-\beta} \right] = \lim_{a_n x \to \infty} \frac{1 - F(a_n x)}{1 - F(a_n(1 + \varepsilon'))} - x^{\beta} =$$

$$= \left(\frac{1 + \varepsilon'}{x} \right)^{\beta} - x^{-\beta} = \frac{(1 + \varepsilon')^{\beta} - 1}{x^{\beta}}.$$

If one takes $(1 + \varepsilon')^{\beta} - 1 = \frac{\varepsilon}{2} x^{\beta}$, that is $\varepsilon' = \left(1 + \frac{\varepsilon}{2} x^{\beta} \right)^{1/\beta}$ the limit will be $\frac{\varepsilon}{2}$.

The limits at left, respectively at right, being 0 and respectively $\frac{\varepsilon}{2}$, it follows that there exists a $n_0(\varepsilon, x) = n_0$ such that for $n > n_0$: $-\varepsilon < n(1 - F(a_n x)) - x^{-\beta} < \varepsilon$ ... $| n(1 - F(a_n x)) - x^{-\beta} | < \varepsilon$ that is $\lim_{n \to \infty} n(1 - F(a_n x)) = x^{-\beta}$.

The reciprocal is completely proved and, besides, from the above mentioned it follows that if $F^n(\alpha_n x + \beta_n) \to e^{-x^{-}}$ there exist positive numbers a_n such that also $F^n(a_n x) \to e^{-x^{-\beta}}$ and hence, by virtue of the Hincin-Feller's lemma: $\frac{a_n}{\alpha_n} \to 1$; $\frac{\beta_n - 0}{\alpha_n} = \frac{\beta_n}{\alpha_n} \to 0$.

The property $\frac{\beta_{[\lambda n]}}{\alpha_{[\lambda n]}} \to 0$ established previously can be therefore extended for any $n\left(\frac{\beta_n}{\alpha_n} \to 0\right)$.

Observations: In the paper of the Annals of Mathematics 1943, page 441, Gnedenko infers from $\frac{a_{[n\beta]}}{a_n} \to \beta^{1/\alpha}$; $\frac{b_{[n\beta]} - b_n}{a_{[n\beta]}} \to 0$ that it is sufficient to take $a_{[n\beta]} = a_n \beta^{1/\alpha}$; $b_{[n\beta]} = b_n$ and denoting $n_s = [n_{s-1} \cdot \beta]$; $n_1 = [n\beta]$, where n is considered as fixed, infers that $a_{n_s} = a_n \beta^{s/\alpha}$; $b_{n_s} = b_n$ for any natural number s and then,making $s \to \infty$ concludes that $\frac{b_{n_s}}{a_{n_s}} \to 0$. The argument is of course erronate. By Gnedenko's proceeding only a particular solution is obtained, and the theorem is demonstrated only for this particular case (only for certain distribution functions F, and not for all possible ones). Nevertheless, the theorem is valid also for the general case, as we have proved. Besides, other solutions can be developed, which don't reduce to those of Gnedenko's. For instance, we can take $\alpha_{n_s} = a_n \cdot \beta^{s/\alpha} (\ln n)^K$. Here $\frac{\alpha_{n_s}}{\alpha_{n_{s-1}}} \to \lambda^{1/\alpha}$ and nevertheless $\frac{\alpha_{n_s}}{a_{n_s}} \to \infty$ and not towards 1, as it should be if Gnedenko's solution represented the general case.

As regards Feller, he does not deal with the general case: $F^n(\alpha_n x + \beta_n) \to e^{-x^{-\beta}}$, but directly with $F^n(a_n x) \to e^{-x^{-\beta}}$, without showing how and why the general case reduces to the latter. Moreover, Feller easily bypasses the reciprocal theorem, stating that there exist positive numbers a_n such that $n(1 - F(a_n)) \to 1$, but does not justify this statement.

If F is a continuous function, it is easy to show that there are numbers a_n such that $n(1 - F(a_n)) = 1$; but if F is not a continuous function, the situation becomes complicated. We have shown, following essentially Gnedenko's proceeding, how to demonstrate the reciprocal.

4. We will now proceed to the second limit law, namely to the case where $F^n(\alpha_n x + \beta_n) \to e^{-(-x)^{\beta}}$; $\to \infty < x \leqslant 0$.

Acting as for § 2 and keeping the same notations we infer

$$F^{[\lambda n]}(\alpha_{[\lambda n]} x + \beta_{[\lambda n]}) \to e^{-(-x)^{\beta}}; \quad F^{[\lambda n]}(\alpha_{[\lambda n+1]} x + \beta_{[\lambda^{n+1}]}) = (F^{[\lambda n+1]}(\alpha_{[\lambda n+1]} x +$$

$$+ \beta_{[\lambda n+1]})) \frac{[\lambda n]}{[\lambda n+1]} \to e^{-\frac{1}{\lambda}\left(-x\right)^{\beta}} = e^{-\left(-\frac{1}{\lambda^{1/\beta}} x\right)^{\beta}}.$$

Applying also here the supplemented Hincin-Feller theorem $\left(A = \frac{1}{\lambda^{1/\beta}},\ B = 0\right)$ it follows:

$$\frac{\alpha_{[\lambda^{n+1}]}}{\alpha_{[\lambda^n]}} \to \frac{1}{\lambda^{1/\beta}};\ \frac{\beta_{[\lambda^{n+1}]} - \beta_{[\lambda^n]}}{\alpha_{[\lambda^n]}} \to 0,\ \text{when } n \to \infty.$$

Denoting $\alpha_{[\lambda^n]} = U_n$; $\beta_{[\lambda^n]} = V_n$, we see in the first place that by virtue of d'Alembert's theorem with respect to series, the series with U_n as general term is convergent because $\frac{U_{n+1}}{U_n} \to \frac{1}{\lambda^{1/\beta}} < 1$ and therefore U_n i.e. $\alpha_{[\lambda^n]} \to$ $\to 0$ when $n \to \infty$. Moreover, let $\varepsilon_1 > 0$ such that $\mu = \frac{1}{\lambda^{1/\beta}} + \varepsilon_1 < 1$; in this case there is a $n_0 = n_0(\varepsilon_1, \lambda)$ such that for $p \geqslant n_0$, $\frac{U_{p+1}}{U_p} < \mu$. Consequently, after a certain p, the terms U_p are decreasing and because $\frac{U_{n_0+1}}{U_{n_0}} <$

$$< \mu;\quad \frac{U_{n_0+2}}{U_{n_0+1}} < \mu; \ldots;\ \frac{U_n}{U_{n-1}} < \mu$$

multiplying there follows: $U_n < \mu^{n-n_0} U_{n_0}$ i.e. $\alpha_{[\lambda^n]} < \left(\frac{1}{\lambda^{1/\beta}} + \varepsilon_1\right)^{n-n_0} \alpha_{[\lambda_{n_0}]}$.

If $n' \geqslant n_0$, then:

$$U_{n'} + U_{n'+1} + \ldots \leqslant U_{n_0} + U_{n_0+1} + \ldots < (1 + \mu + \mu^2 + \ldots)\, U_{n_0} =$$

$$= \frac{1}{1-\mu} U_{n_0} = \eta$$

(η a positive number).

In the second place, as $\frac{V_{n+1} - V_n}{U_n} \to 0$, when $n \to \infty$, considering some $\varepsilon > 0$, there exists a $n_1(\varepsilon, \lambda) \geqslant n_0$ such that for $s \geqslant n_1$:

$$-\frac{\varepsilon}{2\eta} < \frac{V_{s+1} - V_s}{U_s} < \frac{\varepsilon}{2\eta} \therefore -\frac{\varepsilon}{2\eta} U_s < V_{s+1} - V_s < \frac{\varepsilon}{2\eta} U_s.$$

Making $s = n_1, n_1 + 1, \ldots, n - 1$ and adding up, it can be seen that:

$$-\frac{\varepsilon}{2\eta}(U_{n_1} + U_{n_1+1} + \ldots + U_{n-1}) < V_n - V_{n_1} < \frac{\varepsilon}{2\eta}(U_{n_1} + U_{n_1+1} + \ldots)$$

As $U_{n_1} + U_{n_1+1} + \ldots + U_{n1} < U_{n_0} + U_{n_0+1} + \ldots < \eta$ it follows that

$$-\frac{\varepsilon}{2\eta}\eta < V_n - V_{n_1} < \frac{\varepsilon}{2\eta}\eta \therefore -\frac{\varepsilon}{2} < V_n - V_{n_1} < \frac{\varepsilon}{2}.$$

We can draw now several conclusions. In the first place, since $V_{n_1} - \frac{\varepsilon}{2} < V_n < V_{n_1} + \frac{\varepsilon}{2}$ for any $n > n_1$, the sequence $\{V_n\}$ is bounded. In the second place, let us consider some two numbers, m and n, larger than n_1. From $-\frac{\varepsilon}{2} < V_n - V_{n_1} < \frac{\varepsilon}{2}$; $-\frac{\varepsilon}{2} < V_m - V_{n_1} < \frac{\varepsilon}{2}$ it follows: $-\varepsilon < V_m - V_n < \varepsilon$. But then, by virtue of Cauchy's basic theorem, it follows that the sequence $\{V_n\}$ is convergent, since $\varepsilon > 0$ is arbitrary, and m, n have to meet the unique condition $m, n > n_1(\varepsilon)$, λ being fixed.

As a result $\beta_{[\lambda n]} \to x_0$ when $n \to \infty$; x_0 being a finite number representing the limit of this sequence.

In the third place, let us return to the inequality $-\frac{\varepsilon}{2\eta} < \frac{V_{s+1} - V_s}{U_s} < \frac{\varepsilon}{2\eta}$ that is $-\frac{\varepsilon}{2\eta} U_s < V_{s+1} - V_s < \frac{\varepsilon}{2\eta} U_s$, holding for any $s > n_1$ and make $s = n, n+1, \ldots, n+p-1$. Adding up and dividing by U_n we obtain:

$$-\frac{\varepsilon}{2\eta} \cdot \frac{U_n + U_{n+1} + \ldots + U_{n+p-1}}{U_n} < \frac{V_{n+p} - V_n}{U_n} < \frac{\varepsilon}{2\eta} \cdot \frac{U_n + U_{n+1} + \ldots + U_{n+p-1}}{U_n} .$$

Since:

$$\frac{U_{n+1}}{U_n} < \mu ; \frac{U_{n+2}}{U_n} < \mu^2 ; \ldots ; \frac{U_{n+p-1}}{U_n} < \mu^{p-1}$$

it follows from here: $-\frac{\varepsilon}{2\eta}(1 + \mu + \mu^2 + \ldots) < \frac{V_{n+p} - V_n}{V_n} < \frac{\varepsilon}{2\eta}(1 + \mu + \mu^2 + \ldots)$ i.e. $-\frac{\varepsilon}{2\eta} \cdot \frac{1}{1-\mu} < \frac{V_{n+p} - V_n}{V_n} < \frac{\varepsilon}{2\eta} \cdot \frac{1}{1-\mu}$ or else, denoting

$$\frac{\varepsilon}{2\eta(1-\mu)} = \frac{\varepsilon}{2n_0} = \varepsilon' \ , \ -\varepsilon' < \frac{V_{n+p} - V_n}{U_n} < \varepsilon'.$$

Making $p \to \infty$, it follows that $-\varepsilon' \leqslant \frac{x_0 - V_n}{U_n} \leqslant \varepsilon' (n > n_1(\varepsilon))$ and as ε and thus also ε' are arbitrary, it follows that $\frac{x_0 - V_n}{U_n} \to 0$ i.e. $\frac{x_0 - \beta_{[\lambda n]}}{\alpha_{[\lambda n]}} \to 0$ when $n \to \infty$, so that by virtue of Hincin-Feller's theorem: $F^{[\lambda n]}(\alpha_{[\lambda n]} x + \beta_{[\lambda n]}) \to e^{-(-x)^\beta} \Leftrightarrow F^{[\lambda n]}(\alpha_{[\lambda n]} x + x_0) ; \to e^{-(-x)^\beta}$;
after a certain n, the sequence $\{\alpha_{[\lambda n]}\}$ decreasing and $\alpha_{[\lambda n]} \to 0$ when $n \to \infty$.

We can prove now Gnedenko's results, for the general case, using essentially his proceeding.

The theorem of Gnedenko can be stated as follows:

For a distribution function F to be such that $F^n(\alpha_n x + \beta_n) \to e^{-(-x)^\beta}$; $-\infty < x < 0$, it is necessary and sufficient:

1. That a x_0 exists, such that $F(x_0) = 1$ and $F(x_0 - \varepsilon) < 1$ for any $\varepsilon > 0$.

2. $\lim\limits_{x \to -0} \dfrac{1 - F(Kx + x_0)}{1 - F(x + x_0)} = K^\beta$, for any $K > 0$, where $x \to -0$ means $x \to 0$ through values smaller than 0.

The existence of the finite number x_0 was proved before, and the fact that $F(x_0 - \varepsilon) < 1$ is a consequence of the previous statements: $\alpha_{[\lambda^n]} \to 0$; $\beta_{[\lambda^n]} \to x_0$ when $n \to \infty$.

Indeed, for every fixed $x < 0$, $\lim\limits_{n \to \infty} (\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) = x_0$ and thus, beyond a certain n: $\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]} > x_0 - \varepsilon$ so that $F(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) \geqslant (F(x_0 - \varepsilon))$. If $F(x_0 - \varepsilon) = 1$ then, beyond a certain n also $F(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) = 1$ and $F^{[\lambda^n]}(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]})$ would tend towards 1 and not towards $e^{-(-x)^\beta} < 1$. Consequently $F(x_0 - \varepsilon) < 1$.

In view to prove the necessity of condition 2, we start from the fact that:

$$F^{[\lambda^n]}(\alpha_{[\lambda^n]} x + \beta_{[\lambda^n]}) \to e^{-(-x)^\beta} \Leftrightarrow F^{[\lambda^n]}(\alpha_{[\lambda^n]} x + x_0) \to e^{-(-x)^\beta} \Leftrightarrow [\lambda^n](1 -$$

$$- F(\alpha_{[\lambda^n]} x + x_0)) \to (-x)^\beta$$

as well as that beginning from a certain n the sequence $\{\alpha_{[\lambda^n]}\}$ is strictly decreasing, and $\alpha_{[\lambda^n]} \to 0$, when $n \to \infty$.

Let us consider a negative number x, sufficiently near to 0. According to the above mentioned there will exist a finite number s such that: $(-1)\alpha_{[\lambda^s]} \leqslant x < (-1)\alpha_{[\lambda^{s+1}]}$ and hence

$$(-1)\alpha_{[\lambda^s]} + x_0 \leqslant x + x_0 < (-1)\alpha_{[\lambda^{s+1}]} + x_0$$

and alike

$$(-K)\alpha_{[\lambda^s]} + x_0 \leqslant Kx + x_0 < (-K)\alpha_{[\lambda^{s+1}]} + x_0.$$

Of course $s \to \infty$ when $x \to -0$. Consequently:

$$1 - F((-1)\alpha_{[\lambda^{s+1}]} + x_0) \leqslant 1 - F(x + x_0) \leqslant 1 - F((-1)\alpha_{[\lambda^s]} + x_0)$$

$$1 - F((-K)\alpha_{[\lambda^{s+1}]} + x_0) \leqslant 1 - F(Kx + x_0) \leqslant 1 - F((-K)\alpha_{[\lambda^s]} + x_0)$$

and thus:

$$\frac{[\lambda^s]}{[\lambda^{s+1}]} \cdot \frac{[\lambda^{s+1}](1 - F((-K)\alpha_{[\lambda^{s+1}]} + x_0))}{[\lambda^s](1 - F((-1)\alpha_{[\lambda^s]} + x_0))} - K^\beta \leqslant \frac{1 - F(Kx + x_0)}{1 - F(x + x_0)} - K^\beta \leqslant$$

$$\leqslant \frac{[\lambda^{s+1}]}{[\lambda^s]} \cdot \frac{[\lambda^s](1 - F((-K)\alpha_{[\lambda^s]} + x_0))}{[\lambda^{s+1}](1 - F((-1)\alpha_{[\lambda^{s+1}]} + x_0))} - K^\beta$$

when $x \to -0$, $s \to \infty$:

$$\frac{[\lambda^{s+1}]}{[\lambda]^s} \to \lambda;\ [\lambda^s](1 - F((-1)\alpha_{[\lambda^s]} + x_0)) \to 1;$$

$$[\lambda^s](1 - F((-K)\alpha_{[\lambda^s]} + x_0)) \to K^\beta \text{ etc.}$$

so that the left-hand side of the double inequality tends towards $\left(\frac{1}{\lambda} - 1\right)K^\beta$, and the right-hand side towards $(\lambda - 1)K^\beta$.

We have found exactly the same limits as in the case of the double inequality in § 3. The rest of the demonstration is absolutely the same as there, with the single difference that instead of $x \to \infty$, here $x \to -0$.

Consequently: $\lim\limits_{x \to -0} \frac{1 - F(Kx + x_0)}{1 - F(x + x_0)} = K^\beta$, for any $K > 0$.

Condition 2 is wholly justified in the general case. We intend to prove now the reciprocal, i.e. if conditions 1 and 2 of the theorem are met, then there exist positive numbers a_n such that $n(1 - F(a_n x + x_0)) \to (-x)^\beta$, when $n \to \infty$.

In this view we denote by $(-1)a_n$ the smallest $x (x < 0)$ such that:

$$F(x + x_0) \leqslant 1 - \frac{1}{n} \leqslant F(x + x_0 + 0)$$

Such a x exists, because $F(x_0) = 1$ and for any $\varepsilon > 0$, $F(x_0 - \varepsilon) < 1$. Of course $(-1)\ a_n \to 0$ when $n \to \infty$. Hence:

$$F((-1)a_n + x_0) \leqslant 1 - \frac{1}{n} \leqslant F((-1)a_n + x_0 + 0) \leqslant F(-(1 - \varepsilon')a_n + x_0),$$

for any $\varepsilon' > 0$ and thus:

$$n(1 - F(-(1 - \varepsilon')a_n + x_0)) \leqslant 1 \leqslant n(1 - F((-1)a_n + x_0)).$$

Consequently:

$$\frac{n(1 - F(a_n x + x_0))}{n(1 - F((-1)a_n + x_0))} \leqslant \frac{n(1 - F(a_n x + x_0))}{1} \leqslant$$

$$\leqslant \frac{n(1 - F(a_n x + x_0))}{n(1 - F((-1)(1 - \varepsilon')a_n + x_0))}$$

(here x is fixed), that is:

$$\frac{1 - F(a_n x + x_0)}{1 - F(a_n(-1) + x_0)} - (-x)^\beta \leqslant n(1 - F(a_n x + x_0) - (-x)^\beta \leqslant$$

$$\leqslant \frac{1 - F(a_n x + x_0)}{1 - F(a_n(-(1 - \varepsilon')x + x_0))} - (-x)^\beta.$$

Applying the second condition of the theorem, where on the left-hand side instead of K we have $(-x)$, instead of x; $(-a_n)$, and taking into account the fact that when $n \to \infty$; $(-a_n) \to 0$, it can be seen that the left-hand side tends towards $(-x)^\beta - (-x)^\beta = 0$. Equally, in the right-hand side $-(1-\varepsilon')a_n$ appears instead of x, and $a_n x$ instead of Kx, so that instead of K we have $-\frac{x}{1-\varepsilon'}$, and therefore the limit is $\left(-\frac{x}{1-\varepsilon'}\right)^\beta - (-x)^\beta =$ $= \left(\frac{1}{(1-\varepsilon')^\beta} - 1\right) \cdot (-x)^\beta$. If we take $\left(\frac{1}{(1-\varepsilon')^\beta} - 1\right) = \frac{\varepsilon}{2(-x)^\beta}$ that is $\varepsilon' = 1 - \frac{1}{\left(1 + \frac{\varepsilon}{2(-x)^\beta}\right)^{1/\beta}}$ the limit will be $\frac{\varepsilon}{2}$.

From here it follows that starting from a certain sufficiently large n, depending upon ε and x; $-\varepsilon < n(1 - F(a_n x + x_0)) - (-x)^\beta < \varepsilon$. As ε was taken arbitrarly, this means that $\lim_{n\to\infty} n(1 - F(a_n x + x_0)) = (-x)^\beta$ and the reciprocal of the theorem is completely demonstrated.

5. We will proceed now to the third limit case, namely that where: $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}} \Leftrightarrow n(1 - F(\alpha_n x + \beta_n)) \to e^{-x}$.

For the last asymptotic equality to take place, it is clear necessary that $F(\alpha_n x + \beta_n) \to 1$ when $n \to \infty$. We must now take into account the fact that F can be found in one of the two following situations: a) there exists a finite x_0 such that for any $\varepsilon > 0$, $F(x_0 - \varepsilon) < 1$ and at the same time $F(x_0 + 0) = 1$; b) such an x_0 does not exist therefore for every finite y, $F(y) < 1$.

In the first case, to every x corresponds a natural number $n_0(x)$ such that for $n > n_0(x)$: $\alpha_n x + \beta_n \leqslant x_0$ and moreover $(\alpha_n x + \beta_n) \to x_0$ when $n \to \infty$. Particularly, making $x = 0$ and $x = 1$, it follows: $\lim_{n\to\infty} \beta_n = x_0$; $\lim_{n\to\infty} (\alpha_n + \beta_n) = x_0$ and hence:

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}((\alpha_n + \beta_n) - \beta_n) = \lim_{n\to\infty} (\alpha_n + \beta_n) - \lim_{n\to\infty} \beta_n = x_0 - x_0 = 0.$$

Consequently, in this case, starting from a certain n: $\beta_n \leqslant x_0$, $\beta_n \to x_0$ and $\alpha_n \to 0$ when $n \to \infty$. From here also follows that when $x_0 \neq 0$, $\frac{\alpha_n}{\beta_n} \to 0$ when $n \to \infty$.

If $x_0 = 0$, for $n > n_0(x)$: $\alpha_n x + \beta_n \leqslant 0$ and hence for $n > n_0(0)$: $\beta_n \leqslant 0$. Moreover, we will show that starting from a certain n: $\beta_n < 0$. To this end we start from the asymptotic equality $n(1 - F(\alpha_n x + \beta_n)) \to e^{-x}$, and making $x = 0$ we see that $n(1 - F(\beta_n)) \to 1$. From here follows that beyond a certain n: $\beta_n \neq 0$. Let us suppose that this is not true. Then an infinity of natural numbers n_k will exist, for which $\beta_{n_k} = 0$, and for these $n_k(1 - F(\beta_{n_k})) = n_k(1 - F(0))$, so that $\lim_{n_k\to\infty} n_k(1 - F(\beta_{n_k})) = \infty$ if $F(0) < 1$, or 0 if $F(0) = 1$. But any infinite subsequence of a convergent sequence is also convergent and has the same limit as the initial one; hence we must have $n_k(1 - F(\beta_{n_k})) \to 1$. This is a contradiction, and the hypothesis that it doesn't exist a finite number n' such that for $n > n'$; $\beta_n \neq 0$ falls.

Concluding, for $n > max(n_0(0), n')$; $\beta_n < 0$. Let us denote $max(n_0(0), n')$ with n''. If $n > max(n_0(x), n'')$ then $\alpha_n x + \beta_n \leqslant 0$ and $\beta_n < 0$. Particularly, if $x > 0$ and $n > max\,(n_0(x), n'')$: $\alpha_n x - (-\beta_n) \leqslant 0 \Leftrightarrow \frac{\alpha_n}{(-\beta_n)} \leqslant \frac{1}{x}$ and as x is arbitrary, it follows that $\frac{\alpha_n}{\beta_n} \to 0$, when $n \to \infty$. Hence, in the first case $\alpha_n \to 0$, $\beta_n \to x_0$ and $\frac{\alpha_n}{\beta_n} \to 0$, for any finite x_0. In the second case $(\alpha_n x + \beta_n) \to \infty$ when $n \to \infty$, for any finite x and particularly $\beta_n \to \infty$. As regards the behaviour of α_n, we can not say much, but for the ratio $\frac{\alpha_n}{\beta_n}$, yes. Let us consider $x = -y$, with $y > 0$. As $(-\alpha_n y + \beta_n) \to \infty$ when $n \to \infty$, beyond a certain n depending upon y: $-\alpha_n y + \beta_n > 0$; $\beta_n > 0$ and thus $\frac{\alpha_n}{\beta_n} < \frac{1}{y}$.

Consequently $\frac{\alpha_n}{\beta_n} \to 0$ when $n \to \infty$, because y is arbitrary. The fact that both in the first, and in the second case $\frac{\alpha_n}{\beta_n} \to 0$ when $n \to \infty$ was emphasized by Gnedenko (lemma 5 of [4]).

As regards the sequences $\{\beta_n\}$ and $\{\alpha_n + \beta_n\}$ it is the moment to make an observation which will prove to be useful.

In section 1 we have indicated a general proceeding for finding admissible numbers α_n, β_n, corresponding to each of the three possible limit laws. Particularly, taking $U(x) = e^{-e^{-x}}$, $x_1 = 0$, $x_2 = 1$, we obtain as an admissible value for β_n the smallest x satisfying the double inequality: $F(x) \leqslant 1 - \frac{1}{n} \leqslant F(x + 0)$ and for $\alpha_n + \beta_n$ the smallest x satisfying the double inequality: $F(x) \leqslant 1 - \frac{1}{ne} \leqslant F(x + 0)$.

As the smallest number which satisfies the double inequality: $F(x) \leqslant 1 - \frac{1}{n} \leqslant F(x + 0)$ can not be greater than the smallest number which satisfies the double inequality: $F(x) \leqslant 1 - \frac{1}{n+1} \leqslant F(x + 0)$, it follows that $\beta_n \leqslant \beta_{n+1}$ and similarly $\alpha_n + \beta_n \leqslant \alpha_{n+1} + \beta_{n+1}$.

Hence the sequences $\{\beta_n\}$; $\{\alpha_n + \beta_n\}$ can be considered undecreasing without any loss of generality.

The former observation, regarding the first sequence, was already made and used by Gnedenko in the mentioned paper (he based upon his theorem 6).

Further, let us consider a subsequence $\{\psi(n)\}$ of the sequence of natural numbers, the function being such that $\lim_{n\to\infty} \frac{\psi(n)}{n} = \lambda$, a finite number. For instance, $\psi(n) = [\lambda n]$. Then from $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ it follows that

$F^{\psi(n)}(\alpha_{\psi(n)}\ x+\beta_{\psi(n)})\to e^{-e^{-x}}$ and hence $F^n(\alpha_{\psi(n)}\ x+\beta_{\psi(n)})=(F^{\psi(n)}(\alpha_{\psi(n)}\ x+$

$+\beta_{\psi(n)}))\ \ n/\psi(n)\ \to(e^{-e^{-x}})^{1/\lambda}=e^{-1/\lambda\cdot e^{-x}}=e^{-e^{-(x+\ln\lambda)}}$

Consequently the Hincin-Feller lemma is applied, with $A=1$; $B=$ $=\ln\lambda$, so that:

$$\frac{\alpha_{\psi(n)}}{\alpha_n}\to 1;\ \frac{\beta_{\psi(n)}-\beta_n}{\alpha_n}\to\ln\lambda.$$

If $\lambda=1$, α_n; β_n can be replaced by $\alpha_{\psi(n)}$, $\beta_{\psi(n)}$ without any loss of generality, because in this case $F^n(\alpha_{\psi(n)}\ x+\beta_{\psi(n)})\to e^{-e^{-x}}$. If $\lambda\neq 1$, then $\frac{(\beta_{\psi(n)}-\beta_n):\ln\lambda}{\alpha_n}\to 1$, so that by virtue of the Hincin-Feller lemma:

$$F^n(\alpha_n x+\beta_n)\to e^{-e^{-x}}\Leftrightarrow F^n\left(\frac{\beta_{\psi(n)}-\beta_n}{\ln\lambda}\ x+\beta_n\right)\to e^{-e^{-x}}\Leftrightarrow n\left(1-\right.$$

$$\left.-F\left(\frac{\beta_{\psi(n)}-\beta_n}{\ln\lambda}\ x+\beta_n\right)\right)\to e^{-x}$$ Particularly, if $\psi(n)=[\lambda n]$; $\lambda=e$ the necessary and sufficient condition becomes: $n(1-F((\beta_{[ne]}-\beta_n)\ x+\beta_n)\to e^{-x}$ when $n\to\infty$, where β_n is the smallest x which satisfies the double inequality $F(x)\leqslant 1-\frac{1}{n}\leqslant F(x+0)$.

If F is a continuous function, then $F(x+0)=F(x)$ and β_n is the solution of the equation $F(x)=1-\frac{1}{n}$ if the equation assumes a single solution, and respectively the smallest solution of this equation, if the equation assumes several solutions.

If there exists an a_n having a simpler expression than $\beta_{[ne]}-\beta_n$ and such that $\beta_{[ne]}-\beta_n=a_n+0(a_n)$; instead of $\beta_{[ne]}-\beta_n$ we will use a_n because $\frac{\beta_{[ne]}-\beta_n}{a_n}\to 1$. Thus if $\beta_n=(\ln n)^{1/\alpha}$; $\beta_{[ne]}-\beta_n=(1+\ln n)^{1/\alpha}-(\ln n)^{1/\alpha}$ we can take for $a_n=\frac{1}{\alpha}(\ln n)^{(1-\alpha)/\alpha}$ (the main part of the above-mentioned difference), and not $\frac{1}{\alpha}(\ln\ n)^{(\alpha-1)/\alpha}$ as it is indicated in [4] page 445.

Previously we have shown that when α_n, β_n are admissible values, $\frac{\alpha_n}{\beta_n}\to 0$ when $n\to\infty$. Consequently, $\frac{(\beta_{\psi(n)}-\beta_n):\ln n}{\beta_n}\to 0$ i.e. $\frac{\beta_{\psi(n)}}{\beta_n}\to 1$ and particularly $\frac{\beta_{[ne]}}{\beta_n}\to 1$ when $n\to\infty$.

It would be, of course, better to have at one's disposal a simple proceeding which would enable to show the requirements F should meet, without resorting to β_n numbers. Such a simple proceeding, unlike the case of the two limit laws, was found only for some particular cases.

The most important result in this respect is due to von Mieses [6] and can be presented as follows:

Let $x_0(F) = \sup \{x \mid F(x) < 1\}$. If $F(x)$ is a distribution function with $f(x)$ density, positive differentiable on a left neighbourhood of x_0, and $\lim x \uparrow x_0 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-F(x)}{f(x)}\right)$ exists and is equal to 0, then there exist numbers α_n, β_n such that $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ when $n \to \infty$.

Moreover, if we call Von Mieses function $F_X(x)$ corresponding to the random variable X, a function F_X which satisfies the above-mentioned condition, then to each function F such that $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ corresponds a function Von Mieses F_X such that $x_0(F_X) = x_0(F) = x_0$ and $\lim x \uparrow x_0 \frac{1-F(x)}{1-F_X(x)} = 1$ and, conversly, if to a distribution function F corresponds a Von Mieses function F_X meeting the above requirements, then there exist numbers α_n, β_n such that $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ when $n \to \infty$.

This outstanding property was found by A. A. Balkema and L. de Haan [7]. Its practical value is however small, because it is not easy to arrive from F to F_X.

6. It may be adequate to present now briefly the necessary and sufficient conditions developed by Gnedenko [4] and Marcus and Pinsky [5] for F.

Gnedenko starts from his observation that $\frac{\alpha_n}{\beta_n} \to 0$ when $n \to \infty$. He writes: $\alpha_n x + \beta_n = \beta_n\left(1 + \frac{\alpha_n}{\beta_n} x\right)$; introduces a function A defined over the set of β_n numbers as follows: $A(\beta_n) = \frac{\alpha_n}{\beta_n}$ and takes into account the fact that when $\beta_n \to x_0$ (x_0 being defined as above), $A(\beta_n) \to 0$.

As $\{\beta_n\}$ is an undecreasing sequence which tends towards x_0, any z smaller than x_0 and sufficiently near to the latter, ranges between two adequate consecutive terms of this sequence, $\beta_s \leqslant z \leqslant \beta_{s+1}$. Further, Gnedenko extends the definition range of the function A over an interval $[\beta_m, x_0]$, m being any, such that A is a continuous function over this interval, and over each subinterval $[\beta_s, \beta_{s+1}]$ also monotone. In this manner, on each subinterval $[\beta_s, \beta_{s+1}]$; $A(z)$ ranges between $A(\beta_s)$ and $A(\beta_{s+1})$, and $\lim z \uparrow x_0 (A(z)) = 0$.

There exists, of course, an infinity of functions which meet such requirements, e.g. a linear function on each subinterval $[\beta_s, \beta_{s+1}]$ with $A(\beta_s) = \frac{\alpha_s}{\beta_s}$, $A(\beta_{s+1}) = \frac{\alpha_{s+1}}{\beta_{s+1}}$. On such a subinterval $[\beta_s, \beta_{s+1}]$ the function A is increasing if $A(\beta_s) < A(\beta_{s+1})$, decreasing if $A(\beta_s) > A(\beta_{s+1})$, constant if $A(\beta_s) = A(\beta_{s+1})$. From $\alpha_n x + \beta_n = \beta_n\left(1 + \frac{\alpha_n x}{\beta_n}\right)$ one arrive thus to $z(1 + A(z)x)$ and Gnedenko demonstrates that when $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ $\lim z \uparrow x_0 \frac{1-F(z(1+A(z)x))}{1-F(z)} = e^{-x}$, and this for any x; and then also the

reciprocal, namely: if A is a continuous function, defined in a neighbourhood at the left of x_0 with $A(z) \to 0$ when $z \uparrow x_0$ and $\lim z \uparrow x_0 \dfrac{1 - F(z(1 + A(z)x))}{1 - F(z)} = e^{-x}$, then there exist α_n, β_n numbers such that $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$.

If such a function A was known, the criterion could be applied easily, but how to detect such a function? It is why the criterion — notwithstanding its easy theoretical formulation — is not very operational.

As regards Marcus and Pinsky, they started by denoting $1 - F(x) = e^{-g(x)}$ such that $n(1 - F(\alpha_n x + \beta_n)) = ne^{-g(\alpha_n x + \beta_n)} = e^{-g(\alpha_n x + \beta_n) + \ln n}$ and thus: $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}} \Leftrightarrow n(1 - F(\alpha_n x + \beta_n)) \to e^{-g(\alpha_n x + \beta_n) + \ln n} \to e^{-x} \Leftrightarrow$ $\Leftrightarrow g(\alpha_n x + \beta_n) - \ln n \to x$.

They define then $g^{-1}(x) = min\,\{s \mid g(s) \geqslant x\}$ and observe, based on a result found by Gnedenko, that when $g(\alpha_n x + \beta_n) - \ln n \to x$, one can take $\beta_n = g^{-1}(\ln n)$, $\alpha_n = g^{-1}(1 + \ln n) - g^{-1}(\ln n)$. Further they introduce the functions $b(s) = g^{-1}(\ln s)$; $a(s) = g^{-1}(1 + \ln s) - g^{-1}(\ln s)$; $s > 0$, and show that $a(s)$ is function with slow variation, i.e. for any $u > 0$, $\lim\limits_{s \to \infty} \dfrac{a(su)}{a(s)} = 1$. Then, from $g(\alpha_n x + \beta_n) - \ln n \to x$, they infer that $g(\alpha_n x + \beta_n) \sim \ln n + x$ and from here that $g^{-1}(x + \ln n) - g^{-1}(\ln n) \sim x a_n$, proving afterwards that a more general relation: $g^{-1}(x + \ln s) - g^{-1}(\ln s) \sim x a(s)$ holds, i.e. $\lim\limits_{s \to \infty} \dfrac{g^{-1}(x + \ln s) - g^{-1}(\ln s)}{a(s)} = x$.

The above-mentioned explain how one arrived to condition a) from theorem I of Marcus and Pinsky which we will present now, and it will not be difficult to understand how also condition b) was reached.

Theorem I. So that $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$, following conditions are necessary and sufficient:

a) $g^{-1}(x + \ln s) - g^{-1}(\ln s) \sim x a(s) : (s \to \infty)$, where $a(s)$ varies slowly;

b) $g^{-1}(\ln x) = \int_1^x \dfrac{a(s)}{s}\,ds + \varphi(x)$ where $a(s)$ is an arbitrary function with slow variation, and $\dfrac{\varphi(su) - \varphi(s)}{a(s)} \to 0$;

(u a positive number greater than 0).

They give, besides, also a second theorem:

Theorem II. If $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$, then: $g^{-1}(\ln x)$ is a function with slow variation; b) $g^{-1}(\ln x) \sim \int_1^x \dfrac{a(s)}{s}\,ds$, where $a(s)$ is the function with slow variation of theorem I.

If conditions a) and b) of theorem I are met, one could of course take as $a(s)$ the former, that is $a(s) = g^{-1}(1 + \ln s) - g^{-1}(\ln s)$, and as $b(s)$, $g^{-1}(\ln s)$.

We tried to emphasize the mechanism of the results reached by Marcus and Pinsky, so as to show that although they solve, theoretically, completely the problem, their method is not easy, the proceeding which we indicate in the last part of 5, and which represents the simplification of a previous result of Gnedenko, being on the whole more operational, and the imposed condi-

tions more simple. Indeed, it was required that: $n(1 - F((\beta_{[ne]} - \beta_n)x + \beta_n)) \to e^{-x}$, where β_n is the smallest x such that $F(x) \leqslant 1 - \frac{1}{n} \leqslant F(x + 0)$ and once the expression for β_n found, $\beta_{[ne]}$ is obtained by replacing n by $[ne]$.

7. In certain particular cases, simple conditions can be found for the double exponential limit law as well. Thus, if $\alpha_n \to a \neq 0$ (of course $a > 0$) and hence according to § 5, $b_n \to \infty$, a necessary and sufficient condition for $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ is that F should meet the following condition: $\lim_{y \to \infty} \frac{1 - F(ax + y)}{1 - F(y)} = e^{-x}$ for any x.

We will show how this statement is justified. Since $\alpha_n \to a$ and thus $\frac{\alpha_n}{a} \to 1$; $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}} \Leftrightarrow F^n(ax + \beta_n) \to e^{-e^{-x}}$ (the Hincin-Feller lemma), i.e. $n(1 - F(ax + \beta_n)) \to e^{-x}$. As $\beta_n \to \infty$ when $n \to \infty$ for any y at the right of some β_m, there exists a s such that $\beta_s \leqslant y \leqslant \beta_{s+1}$. Consequently: $1 - F(ax + \beta_{s+1}) \leqslant 1 - F(ax + y) \leqslant 1 - F(ax + \beta_s)$ and particularly $1 - F(\beta_{s+1}) \leqslant 1 - F(y) \leqslant 1 - F(\beta_s)$.

Hence:

$$\frac{1 - F(ax + \beta_{s+1})}{1 - F(\beta_s)} \leqslant \frac{1 - F(ax + y)}{1 - F(y)} \leqslant \frac{1 - F(ax + \beta_s)}{1 - F(\beta_{s+1})} \quad \text{i.e.}$$

$$\frac{s}{s+1} \cdot \frac{(s+1)(1 - F(ax + \beta_{s+1}))}{s(1 - F(\beta_s))} \leqslant \frac{1 - F(ax + y)}{1 - F(y)} \leqslant \frac{s+1}{s} \cdot \frac{s(1 - F(ax + \beta_s))}{(s+1)(1 - F(\beta_{s+1}))}$$

Taking into account the fact that when $s \to \infty$; $s(1 - F(ax + \beta_s)) \to e^{-x}$; $s(1 - F(\beta_s)) \to 1$ etc., and when $y \to \infty$, of course also $s \to \infty$, one can see that when $y \to \infty$, the two extreme terms of the double inequality above tend towards e^{-x}, hence also the middle term tends towards the same limit. The necessity of the stated condition is proved.

Let us show now that it is also sufficient.

Here we start from the fact that $F(\beta_n) \leqslant 1 - \frac{1}{n} \leqslant F(\beta_n + 0) \leqslant F(a\varepsilon' + \beta_n)$ with any $\varepsilon' > 0$, a fact from which follows:

$$n(1 - F(a\varepsilon' + \beta_n)) \leqslant 1 \leqslant n(1 - F(\beta_n)), \quad \text{so that:}$$

$$\frac{n(1 - F(ax + \beta_n))}{n(1 - F(\beta_n))} - e^{-x} \leqslant \frac{n(1 - F(ax + \beta_n))}{1} - e^{-x} \leqslant \frac{n(1 - F(ax + \beta_n))}{n(1 - F(a\varepsilon' + \beta_n))} - e^{-x}.$$

When $n \to \infty$, $\beta_n \to \infty$ and the left-hand side tends according to the hypothesis towards $e^{-x} - e^{-x} = 0$. The right-hand side can also be written as follows: $\frac{1 - F(ax + \beta_n)}{1 - F(\beta_n)} : \frac{1 - F(a\varepsilon' + \beta_n)}{1 - F(\beta_n)} - e^{-x}$ and it tends towards e^{-x}:

$e^{-\varepsilon'} - e^{-x} = (e^{\varepsilon'} - 1)\, e^{-x}$. If we take $(e^{\varepsilon'} - 1)\, e^{-x} = \frac{\varepsilon}{2}$ that is $\varepsilon' = \ln\left(1 + \frac{\varepsilon}{2} e^{x}\right)$, the limit for the right-hand side will be $\frac{\varepsilon}{2}$.

From here it follows that there exists a natural number $n_0(\varepsilon)$ such that for $n > n_0(\varepsilon)$: $-\varepsilon < n(1 - F(ax + \beta_n) - e^{-x} < \varepsilon$. As ε is any, it follows: $\lim_{n\to\infty} n(1 - F(ax + \beta_n) = e^{-x}$ The sufficiency of the condition is proved.

Let us now summarize some of the obtained results:

So that there exist numbers α_n and β_n such that:

1°. $F^n(\alpha_n x + \beta_n) \to e^{-x-\beta}$ it is necessary and sufficient that for any $k > 0$:

$$\lim_{x\to\infty} \frac{1 - F(x)}{1 - F(kx)} = k^{\beta}$$

Particularly, if $\lim_{x\to\infty} x^{\beta}(1 - F(x))$ exists and is a finite number different from 0, the above-mentioned condition is satisfied because $\frac{1 - F(x)}{1 - F(kx)} = k^{\beta} \frac{x^{\beta}(1 - F(x))}{(kx)^{\beta}(1 - F(kx)}$, and when $x \to \infty$ both the numerator and the denominator of the latter fraction have the same limit, different from 0, so that $\frac{1 - F(x)}{1 - F(kx)}$ tends towards k^{β}.

It is, of course, possible that $x^{\beta}\ (1 - F(x))$ does not satisfy the above requirement and neverthless $\frac{1 - F(x)}{1 - F(kx)}$ to tend towards k^{β}. It was why we have mentioned that we were dealing with a particular case.

2°. $F^n(\alpha_n x + \beta_n) \to e^{-(-x)^{\beta}}$. In this case there exists a finite x_0 such that $F(x_0) = 1$, $F(x_0 - \varepsilon) < 1$ for any $\varepsilon > 0$ and, moreover, it is necessary that $\lim x \uparrow 0 \frac{1 - F(kx + x_0)}{1 - F(x + x_0)} = k^{\beta}$ for any $k > 0$.

These conditions are necessary and sufficient.

Particularly, if $\lim x \uparrow 0 x^{\beta}(1 - F(x + x_0))$ exists and is a finite number different from 0, the above condition is satisfied, as can be seen, in the same way as for case 1.

3°. $F^n(\alpha_n x + \beta_n) \to e^{-e^{-x}}$ and $\alpha_n \to a \neq 0$. In this case a necessary and sufficient condition is that $\lim_{y\to\infty} \frac{1 - F(ax + y)}{1 - F(y)} = e^{-x}$ for any finite x.

Particularly, if $\lim_{y\to\infty} e^{-\frac{y}{a}}(1 - F(y))$ exists and is a finite number different from 0, the above condition is satisfied.

Thus we have finished our exposal regarding the limit laws for extremal values. In the next paper we will show if, how, and to which extent the obtained results can be used in global statistic control.

REFERENCES

1. THEILER G., TŐVISSI L.: *Valori extremale, repartiția Weibull și unele probleme de control statistic global.* (Extremal Values, Weibull Distribution and Some Problems of Global Statistical Control). Studii și cercetări de calcul economic și cibernetică economică, 4, 1976.
2. FELLER W.: *An Introduction to Probability Theory and Its Applications.* Vol. II, John Wiley and Sons, New York, 1966.
3. GNEDENKO B.W., KOLMOGOROV A.N.: *Grenzverteilungen von Summen unabhängiger Zufallgrössen.* Akademie Verlag, Berlin, 1960.
4. GNEDENKO B.W.: *Sur la distribution limite du terme maximum d'une série aléatoire.* Annals of Mathematics, Vol. 44, no. 3, (VII. 1943)
5. MARCUS M., PINSKY M.: *On the Domain of Attraction of* $e^{-e^{-x}}$. Journal of Mathematical Analysis and Application, Vol. 28, 2, XI, 1969.
6. MIESES VON R.: *La distribution de la plus grande des n valeurs.* Revue de Mathématique de l'Union Interbalkanique, t.I.f.l, pp. 141—160, 1939, or Selected Papers of Richard von Mieses II, American Mathematical Society, Providence, R.I. 1964.
7. BALKEMA A.A., HAAN L.: *On R Von Mieses Condition for the Domain of Attraction of exp.* $(-e^{-x})$. The Annals of Mathematical Statistics, vol. 43, 4, 1972.

26. 09. 1976.

DISTRIBUTIONS INITIALES CORRESPONDANT AUX LOIS LIMITE DES VALEURS EXTREMALES

Les auteurs ont présenté dans un article antérieur, le problème des lois limite pour les valeurs extrémales et ont démontré comment aboutir à ces lois. Il restait à traiter le problème de trouver les distributions initiales F pour lesquelles le maximum M_n, respectivement le minimum m_n d'une sélection de volume n tend vers l'une des trois lois limite trouvées, lorsque $n \to \infty$. Dans le présent ouvrage, les auteurs seront intéressés tout spécialement par les distributions initiales qui mènent aux distributions limite du type Weibull afin de mieux savoir dans quelle mesure la distribution Weibull peut-être utilisée dans le problème de statistique qui les préoccupe.

Les auteurs vont commencer par un cas particulier de la lemme de Feller [2], de la lemme de Hincin-Feller, pour mieux dire, car Hincin est celui qui avait découvert la propriété asymthotique des distributions du même type (voir, par exemple [3], p. 39), et Feller lui a donné la forme finale qu'ils ont utilisée dans [1] et qu'ils vont utiliser maintenant aussi, car elle rend inutiles certaines lemmes supplémentaires (par exemple les lemmes 1 et 2 du § 3 de [4], la première étant présentée aussi dans [3] page 41).

Ils mettrons au point quelques démonstrations se trouvant dans la littérature de spécialité concernant certaines distributions F; ils trouveront quelques résultats nouveaux et présenteront sommairement d'autres résultats relativement récents.

ПЕРВОНАЧАЛЬНОЕ РАСПРЕДЕЛЕНИЕ ДЛЯ ПРЕДЕЛА ЭКСТРЕМАЛЬНЫХ ЗНАЧЕНИЙ

В одной из предшествующих статей [1] мы показали как ставися вопрос и как познаются законы вероятного ограничения для экстремальных значений. Остался открытым вопрос нахождения первоначальных распределений F, для которых максимум M_n. соответственно минимум m_n селекции объема n стремится к одной из трех законов найденного предела, в то время как $n \to \infty$. В частности нас будут интересовать первоначальные распределения, которые приводят к предельным распределениям типа Вейбул, для того, чтобы можно было лучше дать себе отчет в какой мере распределение Вейбул может быть использовано в интересующих нас вопросах статистики.

Начнем с частного случая леммы Феллера [2], лучше сказать, леммы Хиннчин-Феллер, поскольку Хинчин раскрыл асимптотические свойства распределений этого типа (см., например [3], стр. 39), а Феллер придал окончательную форму, которую мы использовали в [1] и будем использовать и теперь, так как использование дополнительных лемм будет бесполезным (например, лемы 1 и 2 параграфа 3 из [4], перевая лемма представлена и в [3] на странице 41).

В работе будут прнведены а порядок некоторые демонстрации, отраженные в литературе по специальности, касающейся некоторых распределений F; будут расширены одни результаты и кратко представлены другие, более новые.