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Queueing Systems with Various Service Intensities

The authors analyse the particular case (for $s = 2$) of the $M/M/S$ model with s service stations with various service parameters. We propose to elaborate a method which should facilitate calculations and increase the exactness of application.

In most cases, the queueing models have been analysed by taking into account several service stations in parallel, with service intensities equal for all stations. In reality, this hypothesis takes place seldom. If apparently, this problem may be neglected, for example, in case of pay desks in department stores, in case of production processes with a different technicity and oldness of the number of machines, the $M/M/S$ model with equal service intensities no longer corresponds.

In such cases, one cannot work with the simplified acceptance of the equal service intensities of all stations.

The aim of this article is to analyse the $M/M/S$ queueing system with $S = 2$ service stations in parallel, with the service intensities μ_1, respectively μ_2; $\mu_1 \neq \mu_2$. The schematic model appears as in figure 1.

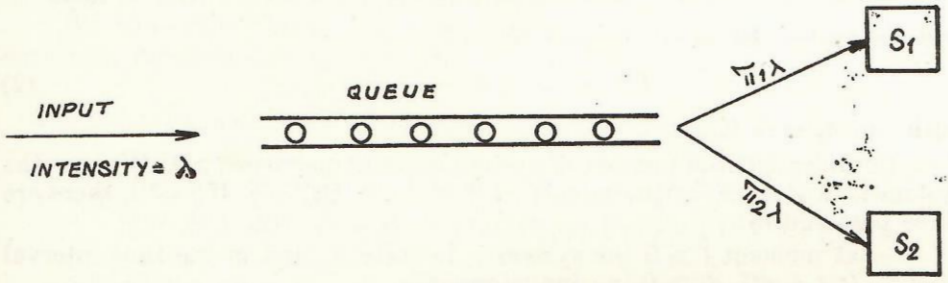

Fig. 1

We consider that a unit arrives at an queueing system at a moment when there is no other unit in the system. This unit has thus the freedom of choosing between the two service stations and one will decide with the probability π_i for station S_i, $i = 1, 2$. We thus have:

$$0 \leqslant \pi_i \leqslant 1, \qquad i = 1, 2.$$

$$\sum_{i=1}^{2} \pi_i = 1.$$

The following situations may appear:

a) $\mu_1 > \mu_2$ and $\pi_1 > \pi_2$.

In this case, some of the customers knowing the service intensities of the stations have preferences for the stations with a more rapid service.

b) $\mu_1 > \mu_2$ and $\pi_1 < \pi_2$.

In this situation, we have a choice in which customers prefer, for example, the quality of services and not the rapidity of solution.

The extreme case $\pi_1 = 1$ (respectively $\pi_2 = 1$) represents the situation in which all customers decide on station S_1 (respectively S_2), if they have possibilities of choice.

THE STATE EQUATIONS OF THE MODEL

The state equations are obtained similarly with the $M/M/S$ model. The problem arises of distinguishing additional cases which correspond to the choice of stations.

We denote:

P_{10} — the probability that a unit is in the system in the S_1 service station;

P_{01} — the probability that a unit is in the system in the S_2 station.

Let $X(t)$ be the number of units entering the system in the time interval $(0, t]$ and $Y(t)$ — the number of the served units leaving the system in the interval $(0, t.]$ It is known that in the case of the system with a single station the probability $p[X(\tau) = 1]$ for a unit to enter the system in the time interval $(0, \tau]$ is given by:

$$P[X(\tau) = 1] = \mu\tau + \tau\theta_1(\tau) \tag{1}$$

where $\lim_{\tau \to 0} \theta_1(\tau) = 0$, and that the probability for a served unit to abandon the system in the same interval $(0, \tau)$ is:

$$P[Y(\tau) = 1] = \mu\tau + \tau\theta_2(\tau) \tag{2}$$

with $\lim \theta_2(\tau) = 0$.

Consider $\xi(t)$ the number of system units at moment $t \geqslant 0$. We say the system is in state E_j at the moment t if we have $\xi(t) = j$. If $j = 0$, there are three possibilities:

— at moment $t > 0$ the system is in state E_0 and in the time interval $(t, t + dt]$, $dt > 0$, no input appears;

— at moment $t > 0$, the system in state E_1 and in the time interval $(t, t + dt]$, $dt > 0$, no unit arrives and the existent unit is served by station S_1;

— at moment $t > 0$ the system is in state E_1 and in the interval $(t, t + dt]$, $dt > 0$ no unit arrives and the existent unit is served by station S_2.

We thus have:

$$p_0(t + dt) = (1 - \lambda dt)\, p_0(t) + \mu_1 P_{10}(t) + \mu_2 P_{01}(t). \tag{3}$$

We now consider case $j = 1$. This unit may be found in the system at station S_1 with the probability p_{10}, or may be found in the system at station S_2 with the probability P_{01}. We shall explain the case when the unit is served by the first station. The following situations may exist:

— at moment t, $(t > 0)$, the system is in state E_1 and in the time interval $(t, t + dt)$, $dt > 0$ no input into and no output from the station takes places;

— at moment t, $(t > 0)$, the system is in state E_0 and in the interval $(t, t + dt)$, $dt > 0$, a unit enters the system and no start takes place;

— at moment t, $(t > 0)$, the system is in state E_2 and in the time interval $(t, t + dt]$, the unit is served by station S_1 (the unit leaving the system immediately) and no unit enters the system.

Thus, we obtain:

$$P_{10}(t + dt) = [1 - (\lambda + \mu_1)\, dt]P_{10}(t) + \mu_2 P_2(t)\, dt + \pi_1 \lambda P_0(t)\, dt. \tag{4}$$

When $j = 1$, but the unit chooses station S_2, we shall obtain by a similar reasoning, equation:

$$P_{01}(t + dt) = [1 - (\lambda + \mu_2)\, dt]p_{01}(t) + \mu_1 P_2(t)\, dt + \pi_2 \lambda P_0(t)\, dt. \tag{5}$$

For $j \geqslant 2$, the state equations coincide with those of the model $M/M/S$, that is we shall have:

$$p_i(t + dt) = [1 - (\lambda + \mu)\, dt]\, p_j(t) + \mu P_{j+1}(t)dt + \lambda P_{j-1}(t)\, dt. \tag{6}$$

where: $\mu = \mu_1 + \mu_2$:

$$P_1(t) = P_{10}(t) + P_{01}(t). \tag{7}$$

We have thus obtained as state equations of the model with two stations with different service intensities, the following:

$$n = 0: \quad p_0(t + dt) = (1 - \lambda dt)\, p_0(t) + \mu_1 p_{10}(t)\, dt + \mu_2 P_{01}(t)\, dt$$

$$n = 1: \begin{cases} P_{10}(t + dt) = [1 - (\lambda + \mu)\, dt]\, P_{10}(t) + \mu_2 P_2(t)dt + \pi_1 \lambda P_0(t)dt \\ P_{01}(t + dt) = [1 - (\lambda + \mu)\, dt]\, P_{01}(t) + \mu_1 P_2(t)\, dt + \pi_2 \lambda P_0(t)\, dt \end{cases} \tag{8}$$

$$n \geqslant 2: \; P_j(t + dt) = [1 - (\lambda + \mu)\, dt]\, P_j(t) + \mu P_{j+1}(t)\, dt + \lambda P_{j-1}(t)dt.$$

If the phenomenon is stationary ($P_n(t) \to P_n$), we shall obtain the following state equations:

$$\lambda P_0 = \mu_1 P_{10} + \mu_2 P_{01}$$

$$(\lambda + \mu_1) P_{10} = \mu_2 P_2 + \pi_1 \lambda P_0$$

$$(\lambda + \mu_2) P_{01} = \mu_1 P_2 + \pi_2 \lambda P_0 \tag{9}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\lambda + \mu) P_n = \mu P_{n+1} + \lambda P_{n-1}' \qquad n \geqslant 2.$$

By introducing the traffic intensity:

$$\psi = \frac{\lambda}{\mu} = \frac{\lambda}{\mu_1 + \mu_2} \tag{10}$$

and marking the corresponding calculations, we obtain:

$$P_{10} = \frac{\psi}{1 + 2\psi} \cdot \frac{\mu}{\mu_1} \cdot (\psi + \pi_1) P_0$$

$$P_{01} = \frac{\psi}{1 + 2\psi} \cdot \frac{\mu}{\mu_2} (\psi + \pi_2) P_0$$

$$P_1 = \frac{\psi}{1 + 2\psi} \cdot \frac{\mu}{\mu_2} \left(1 + \frac{\mu}{\mu_1} + \frac{\mu_2 - \mu_1}{\mu_1} \pi_1\right) P_0$$

$$P_2 = \frac{\psi^2}{1 + 2\psi} \cdot \frac{\mu}{\mu_2} \left(1 + \frac{\mu}{\mu_1} \psi + \frac{\mu_2 - \mu_1}{\mu_1} \pi_1\right) P_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_n = \frac{\psi^n}{1 + 2\psi} \cdot \frac{\mu}{\mu_2} \left(1 + \frac{\mu}{\mu_1} + \frac{\mu_2 - \mu_1}{\mu_1} \pi_1\right) P_0, \quad n > 2. \tag{11}$$

From state $\sum_{n=0}^{\infty} P_n = 1$, we obtain for p_0 expression

$$P_0 = \frac{1 - \psi}{1 + \frac{\psi}{1 + 2} \cdot \frac{\mu_1}{\mu_2} \left[1 + \left(1 + \frac{\mu_2^2}{\mu_1^2}\right) \psi - \left(1 - \frac{\mu_2^2}{\mu_1^2}\right) \pi_1\right]}. \tag{12}$$

Thus, the model is completely defined because all probabilities p_n depend on p_0.

To obtain, for example, the probability for expectation $p(W > 0)$ to exist in the system, we notice that:

$$P(W > 0) = P(n > 2) - 1 - (P_0 + P_{10} + P_{01}). \tag{13}$$

By introducing denotations:

$$\alpha = 1 + \frac{\mu}{\mu} + \frac{\mu_2 - \mu}{\mu_1}\pi_1$$

$$\beta = \frac{\mu_2}{\mu_1}\left(1 + 2\psi\right) + \psi\left[1 + \left(1 + \frac{\mu_2^2}{\mu_1^2}\right)\psi - \left(1 - \frac{\mu_2^2}{\mu_1^2}\right)\pi_1\right]. \tag{14}$$

$$\gamma = 1 + \frac{\psi}{1 + 2\psi} \cdot \frac{\mu_1}{\mu_2}\left[1 + \left(1 + \frac{\mu_2^2}{\mu_1^2}\right)\psi - \left(1 - \frac{\mu_2^2}{\mu_1^2}\right)\pi_1\right]$$

we obtain:

$$P(W > 0) = \psi^2 \frac{\mu}{\mu_1} \cdot \frac{\alpha}{\beta}. \tag{15}$$

From relations:

$$\bar{n} = \sum_{n=0}^{\infty} nP_n, \quad \bar{\gamma} = \sum_{n=3}^{\infty} (n-2)\,P_n \tag{16}$$

we shall find:

$$\bar{n} = \frac{\psi}{\psi - 1} \cdot \frac{\mu}{\mu_1} \cdot \frac{\alpha}{\beta}; \quad \bar{\gamma} = \frac{\psi^3}{1 - \psi} \cdot \frac{\mu}{\mu_1} \cdot \frac{\alpha}{\beta}. \tag{17}$$

Other characteristics of the model can be obtained in a similar way.

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MODELE D'ATTENTE A INTENSITES DIFFERENTES DE SERVICE

Les auteurs font l'analyse du cas particulier (pour $s = 2$) du modèle $M/M/S$ ayant s stations de service à paramètres de service différents. Les auteurs se proposent d'élaborer une méthode qui puisse faciliter les calculs et accroître la précision de l'application.

МОДЕЛЬ МАССОВОГО ОБСЛУЖИВАНИЯ РАЗЛИЧНОЙ ИНТЕНСИВНОСТИ

Авторы анализируют частный случай (для $s = 2$) модели M/MS при $s = 2$ пунктам обслуживания с раздичными параметрами обслуживания. Мы намериваемся разработать метод, позволяющий облегчение вычисления и увеличения точности применения.

26. XII. 1975