CS. FABIAN
R. ŞERBAN
Department of Economic Cybernetics
Academy of Economic Studies

A Comparative Study of "Cutting Plane" Methods in Convex Nonlinear Mathematical Programming

The study presents comparative results of some "cutting plane" type methods for a few test problems of nonlinear programming known in literature.

The methods of Kelley [1], Kleibohm [2] Zoutendijk [3], Wolfe [4] and a modified variant of Kelley's method have been tested.

1. INTRODUCTION

Many economic problems can be modelled and solved as mathematical programming problems. Most often, the complexity of the economic phenomena determines the nonlinear character of the models. In solving these models, the problem arises of using the most adequate known methods and the elaboration of some special methods for surely obtaining the seeked solution with the desired exactness. In this context, the comparative studies on the efficiency of mathematical programming algorithms are extremely useful.

The present work makes a comparative study of several "cutting plane" type methods. The methods of Kelley (KOR), Kleibohm (KEK), Zoutendijk (KEZO), Wolfe (KEWO) and a modification of Kelley's method (KEMO) have been programmed and tested.

2.1. Description of the algorithm

Consider the convex nonlinear programming problem
$min\, f(x)$, with constraints

$$g_j(X) \leqslant 0, \qquad j = 1, ..., m \tag{1}$$
$$X \geqslant 0$$

where $f(X)$ and $g_j(X)$, $j = 1, ..., m$, are convex functions

$$f: E^n \longrightarrow E, \qquad g_j: E^n \longrightarrow E,\ X \in E^n,$$

where E^n is the $-\ n$ dimensional Euclidean space.

Problem (1) can always be reduced to a problem in which the efficiency function is linear so that:

$$min\ x_{n+1}$$
$$f(X) - x_{n+1} \leqslant 0 \tag{2}$$
$$g_j(X) \leqslant 0 \qquad j = 1, ..., m$$
$$X \geqslant 0, \qquad x_{n+1} \in E.$$

Obviously, if $X^* \in E^n$ is the optimum solution of problem (1) then, point $(X^*, x^*_{n+1}) \in E^{n+1}$, with $x^*_{n+1} = f(X^*)$, is the optimum solution for problem (2) and conversly.

The convergence conditions of the algorithm are the followings

a) The set of admissible solutions

$$M = \{X \mid g_j(X) \leqslant 0, \qquad j = 0, ..., m\},$$

where $g_0(X) = f(X) - x_{n+1}$
is contained in a bound polyhedron P°.

$P^\circ = \{X \mid AX \leqslant b\}$, and A is a matrix of $p \times n$ size where $p \leqslant m$.

b) Functions $g_j(X)$ are differentiable in P° and there is a constant value G, so that

$$\| g_j(X) \| \leqslant G, \qquad \text{for any } X \in P^\circ, \qquad j = 0, ..., m.$$

The algorithm consists in reducing the solution of the convex programming problem (2) to the solving of a number of linear programming problems:

$$min\ CX \tag{3}$$
$$X \in P^i$$
$$X \geqslant 0.$$

Let X^i be the solution of problem (3) at iteration i (Start with $i = 0$).

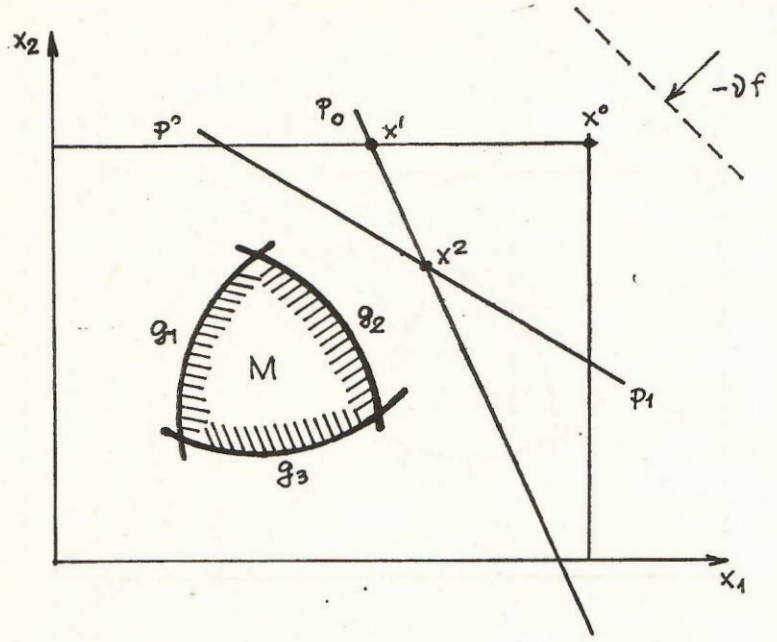

Fig. 1

If $X^i \in M$, then X^i is the optimum solution.

If $X^i \notin M$, determine

$$g_k(X^i) = \max_{0 \leqslant j \leqslant m} g_j(X^i)$$

and a cutting hyperplane is built from point X^i (corresponding to constraint $g_k(X)$ of equation:

$$p_i(X) = g_k(X^i) + (X - X^i)\nabla g_k(X^i) = 0$$

and the new polyhedron P^i is

$$P^{i+1} = P^i \cap \{ X \mid g_k(X^i) + (X - X^i)\nabla g_k(X^i) \leqslant 0 \}.$$

Assign $i := i + 1$ and solve again problem (3).

In case $n = 2$ a suggestive image of the algorithm in figure 1 is presented and, where:

X° is the solution of problem (3) corresponding to polyhedron P° and p_0 the secant plane corresponding to X°; X^1 is the solution of problem (3) corresponding to P^1 and so on.

2.2. Considerations of calculation

In [1], Kelley makes the following remarks:

a) Except for some special functions, the point will never be in the field and consequently, the stopping condition:

$$g_j(X^k) \leqslant \varepsilon \qquad (j = 0, 1, ..., m)$$

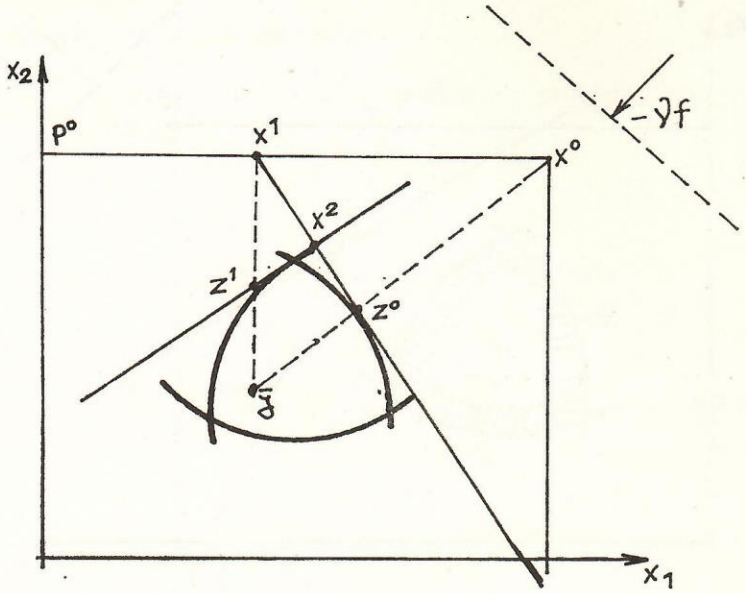

Fig. 2

is not universally valid. It can be satisfied and yet point X^k be far from the optimum point X^*.

b) At the end of the optimization process, the cutting planes $p_i(X)$ tend to become parallel and degenerations appear in the linear programming problem (3).

c) Convergencies may appear in wrong solutions or oscillations.

3. OTHER "CUTTING PLANE" TYPE METHODS

3.1. **The Kleibohm method** [2], [11] requires, in addition to Kelley's method, the knowing of an interior point $\bar{y} \in \mathring{M}$.

The difference against Kelley's method consists in the building of the cutting plane, namely Kleibohm builds hyperplanes tangential to M, in Z^i, Z^i being the points where the straight line segment $[\bar{y}, x^i]$ touches the frontier of M. $(i = 0, 1, ...)$.

In the case of two variables the algorithm can be illustrated by figure 2.

3.2. **The Zoutendijk method** [3] is similar to Kleibohm's method. The difference consists in the fact that the interior point is not fixed but is modified from an iteration to another.

Denoting (at iterration i)

y_i — the interior point $(y_i \in \mathring{M})$

Z_i — the point on the frontier,

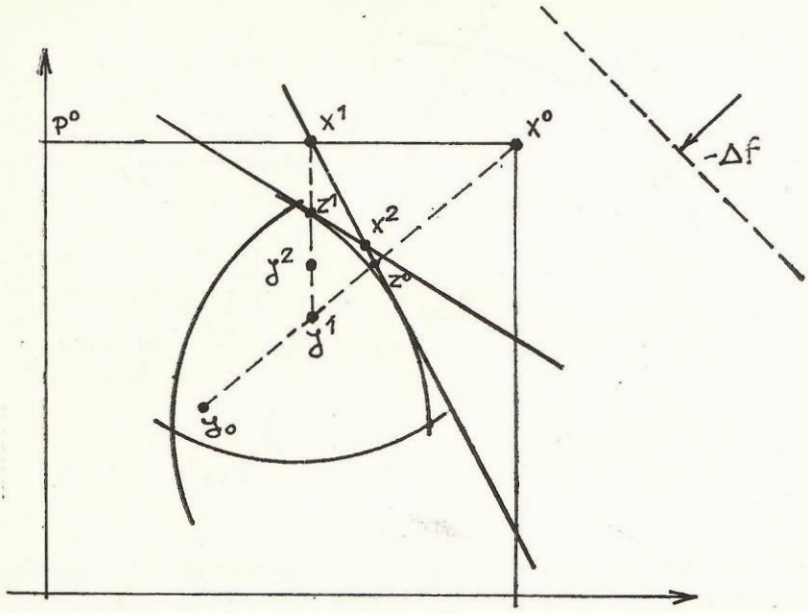

Fig. 3

then the new interior point y^{i+1}, at iteration $(i+1)$ will be a point on segment $[y^i, Z^i]$;

$$y^{i+1} = \tau y^i + (1 - \tau) Z^i, \qquad \text{where} \qquad \tau \in (0, 1).$$

In his algorithm, Zoutendijk takes the value $\tau = \frac{1}{2}$.

Graphically, the algorithm is illustrated in figure 3.

It is noticed that, point y° being given, the new interior point y^1 is determined in the middle of segment $[y^\circ, Z^\circ]$ and so on.

3.3. **The Wolfe Method** [4]. Ph. Wolfe proposes that at each iteration "k", a new point, respectively a new cutting hyperplane should be built in addition to Kelley's method, as follows:

$$X^{k+1} = \sum_{i=0}^{k} X^i V^i$$

where v^i are the values of the dual variables attached to the constraints (to the cutting planes) generated with the points

$$X^i (i = 0, 1, ..., k).$$

This new point is suggested by the Kuhn-Tucker conditions, and by the fact that for a quadratic programming problem, the algorithm is finite.

3.4. **The Modified Kelley Method.** A modification of Kelley's method has, also been tested. The modification consists in the fact that at each iteration two

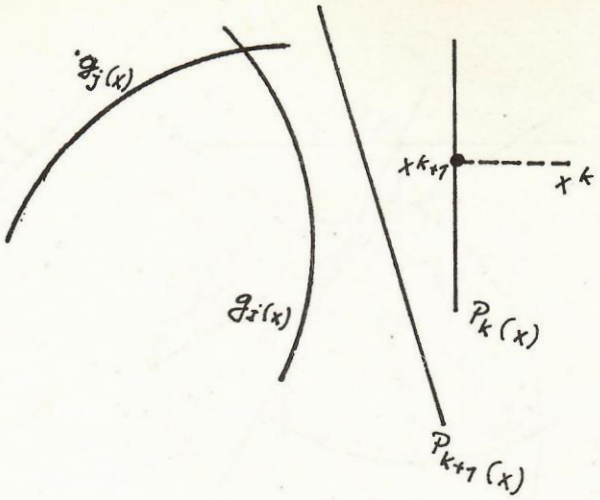

Fig. 4

secant hyperplanes are built. By knowing at iteration k point X^k, the new X^{k+1} is established as being the projection of point X^k on the cutting hyperplane $p_k(X)$ corresponding to X^k. Thus, from X^{k+1} a new cutting hyperplane is built $p_{k+1}(X)$, as seen in figure 4.

3.5. **Other Aspects.** M. Curtillot [5] gives a suggestion for generating the initial polyhedron P° and underlines the possibility of building an infinity of hyperplanes for a given point X^k.

Curtillot also proposes the constant preservation of the dimensions of the linear programming problem (3).

In [6], Eaves and Zangvill give an answer to the question when a hyperplane can be eliminated from the next calculations in problem (3). Thus, denoting by $Z(X)$ the value of the objective function in problem (3), if

$$Z(X^{k+1}) \leqslant Z(X^i) - \delta(X^i), \qquad i = 0, 1, ..., k$$

where

$$\delta(X^i) = \sum_{j=0}^{m} g_j^2(X^i),$$

then, the cutting hyperplane generated by points X^i can be eliminated.

In [7], Veinott gives a generalization of the cutting plane method for unimodal functions and Topkis [8] gives a cutting plane method in which the consecutive polyhedrons P°, P^i, ... are not included one into the other.

L. Spircu [9] presents the method and a general block scheme for the programming of Kelley's method.

Cs. Fabian [10] gives a Kelley type algorithm combined with Korte's lexicographical algorithm for integer nonlinear programming.

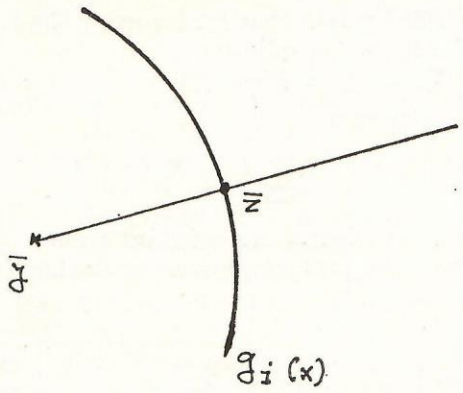

Fig. 5

4. NUMERICAL TESTS

4.1. Establishing an interior point and a point exterior to field M. Consider an arbitrary point $X \in E^n$ (for example the order of the system of coordinates).

a) If X is interior ($\bar{y} := X$), then an exterior point $\bar{X}$ is established as follows: a certain constraint $g_j(X)$ is chosen and a point $\bar{Z}$ on the frontier of $g_j(X)$ is established (figure 5).

Then, the exterior point X is established as follows:

$$\bar{X} = \bar{y} + \lambda(\bar{z} - \bar{y}) \qquad \text{with} \quad \lambda > 1.$$

b) If X is an exterior point ($\bar{X} := X$), then an interior point $\bar{y}$ can be established [2] as follows:

Define $\delta_j = \begin{cases} 1, & \text{if } g_j(\bar{X}) \geqslant 0 \\ 0, & \text{if } g_j(\bar{X}) < 0 \end{cases}$

$$j = 0, 1, \ldots, m.$$

Consider $\bar{\xi} = \max_j g_i(\bar{X})$.

Then point $\left(\dfrac{\bar{X}}{\xi}\right)$ is an interior point for problem

$$\min \xi$$

$$\begin{aligned} & g_j(X) - \delta_j \xi \leqslant 0, \\ & X \geqslant 0, \quad \xi \in R, \end{aligned} \qquad j = 0, 1, \ldots, m. \tag{4}$$

When, in the course of minimization, the first negative value for ξ is obtained, then point $\bar{y} := X$ is a point interior to M.

4.2. **Establishing the initial polyhedron** P°. By using the exterior point $\overline{X} \in M$, polyhedron P° is established as follows:

$$P^\circ = \{ X \mid g_j(\overline{X}) + (X - \overline{X}) \nabla g_j(\overline{X}) \leqslant 0, \qquad j = 0, 1, ..., m \}.$$

For P° to be bound, constraint

$$\sum_{i=1}^{n} x_i = K$$

is added, where K is a sufficiently big constant value.

4.3. **Test Problems.** Nine test problems known as the Lawler and Bell problems [12] have been used:

No.	Problem	No. var.	No. constraint	Objective Function	Kind of constraint	f(x*)
1	LB1	5	11	$f = \sum_{i=1}^{5} x_i^2$	linear	7.44
2	LB2	7	12	$f = x_1x_7 + 3x_2x_6 + x_3x_5 + 7x_4$	10 linear. 2 quadrat	0.00
3	LB3	7	12	$f = \sum_{i=1}^{7} x_i$	„	7.33
4	LB4	7	12	$f = \sum_{i=1}^{7} \sum_{j=1}^{7} x_ix_j - \sum_{i=1}^{7} x_i^2$	„	16.00
5	LB5	7	12	$f = \sum_{i=1}^{7} x_i^2$	„	13.68
6	LB6	8	15	$f = 3x_1x_4 + x_2 + x_2x_5 + x_2x_7 + 6x_3x_8 + x_5x_7 + 11x_6$	13 linear 2 quadrat	9.47
7	LB8	8	15	$f = \sum_{i=1}^{8} x_i^2$	„	39.20
8	LB9	8	15	$f = \sum_{i=1}^{8} x_i$	„	12.65
9	LB10	9	15	$f = x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_2x_5x_7 + x_6x_7x_8$	2	0.00

Remark: In problems LB2—LB10, as compared to [12], two nonconvex constraints each have been eliminated (the nonconvex constraints were not taken into consideration in the tests made).

4.4. **Results of the tests.** The significance of the denotations in table 1 with results is the following:

T — the calculation time, in seconds (established with a subroutine characteristic of the time calculus);

IT — the number of interations (the number of appeals of the SIMPLEX subroutine);

NE — the number of estimations of function and gradient determined by formula:

$$NE = M1 \cdot NF + N \cdot NG.$$

where: NF represents the number of appeals of the FUNC subroutine (for calculating the values of function);
NG — the number of appeals of GRAD subroutine (for the calculus of gradients);
M1 = the number of constraints;
N = the number of variables.

The calculations were made with precision $\varepsilon = 0.01$ both in the main programme and in the simplex subroutine. The results denoted by (*) are obtained with $\varepsilon = 0.02$.

In PCTRES subroutine, for establishing the point on constraints (necessary in the Kleibohm and Zoutendijk programmes) precision $\varepsilon_1 = 0.001$ was used.

Sign (+) used in the results obtained with the Wolfe program shows that the first four iterations are Kelley iterations, Wolfe's formula going to be applied only afterwards (because of the strong degeneration of the dual problem of (3)).

Sign (?) represents the fact that for the respective problems, a cycle to a suboptimal solution was obtained. (This may happen due to the points chosen-or determined) and due to the considered precision.

The results of the tests are presented in table 1. The exterior point for all the results presented was the origin of the axes of co-ordinates. Tests were also made in which the exterior point $x_j = b_j + 1 (j = 1, ..., n)$ where b_j is the upper limit of variable x_j.

Table 1

No.	Problem	Kelley-original (KOR)			Kelley-modified (KEMO)			Wolfe (+) (KEWO)			Kleibohm (KEK)			Zoutendijk (KEZO)		
		T	IT	NE	T	IT	NE	T	IT	NE	T	IT	NE	T	IT	NE
1	L-B-1	5.87	12	288	7.04	10	441	10.03	11	407	13.8	11	2127	17.8	11	2042
2	L-B-2	2.70	1	137	2.80	1	157	2.68	2	137	6.6	4	892	4.3	1	384
3	L-B-3	2.30	0	117	2.30	0	117	2.32	1	117	2.7	1	130	2.4	0	130
4	L-B-4	2.77	1	137	2.96	1	157	2.74	2	137	8.07	5	1250	11.9	5	1478
5	L-B-5	17.3*	35	817	16.2*	22	1017	45.0*	34	1377		?			?	
6	L-B-7	3.53	1	168	3.95	1	208	3.88	1	184	5.05	1	488	5.65	2	504
7	L-B-8	10.18*	20	456	12.0*	14	856	23.9*	18	952		?		29.2	13	3600
8	L-B-9	3.03	0	152	3.15	0	160	3.22	0	160	3.27	0	176	3.28	1	176
9	L-B-10	3.82	2	184	4.2	1	208	3.80	1	184	4.99	1	456	5.60	2	472

The calculus time is proportional to the number of estimations of function and gradients (NE) and to the number of iterations.

For the tests made, "the classification" of algorithms, by taking into consideration denotation NE places Kelley's algorithm first, followed by Kelley modified and Wolfe. The big number of function estimations for establishing the points on the frontier places the Kleibohm and Zoutendijk algorithms after the others.

Observation. In the present stage, the Kleibohm and Zoutendijk programs contain still many intermediate outputs, a fact which has to some extent increased the calculus time.

5. REMARKS AND COMMENTS

a) The hypothesis of bounding field M is still hard to check. In practical problems, however, there are very often upper and lower bounds for variables

$$a_i \leqslant x_i \leqslant b_i, \qquad (i = 1, ..., n).$$

b) The hypothesis of bounding P_0 may be obtained for practical problems with a supplementary constraint of the form

$$\sum_{i=1}^{n} x_i \leqslant K$$

where $K = \sum_{i=1}^{n} b_i$.

c) Just like with any nonlinear programming problem, with these methods too, problems linked with the exactness of calculations appear towards the end optimization, suboptimal solutions being even obtained occasionally.

d) With Kleibohm's method, the interior point $\bar{y}$ should not be close to the frontier and to the optimum point. It may eventually be obtained through P. Huard's method of centres.

e) With Zoutendijk's method, for the interior point should not get close to the frontier rapidly, it is necessary to use a value $\tau < \frac{1}{2}$ or the value of τ should be changed during iterations.

REFERENCES

1. KELLEY J. E.: *The Cutting Plane Method for Solving Convex Programs.* J. Soc. Ind. Appl. Math., Vol. 8, 1960 (SIAM J.).
2. KLEIBOHM K.: *Ein verfahren zur approximativen Lösung von Konvexen Programmen.* Diss. Zürich, 1966.
3. ZOUTENDIJK G.: *Nonlinear Programming: a Numerical Survey.* SIAM J. on Control, Vol. 4, no. 1, 1966.
4. WOLFE PH.: *Accelerating the Cutting Plane Method for Nonlinear Programming.* SIAM J., Vol. 9, No. 3, 1961.
5. CURTILLOT M.: *Deux algorithmes de programmation mathématique.* RAIRO, 7, 1973, (Mathématiques), R-1.

6. EAVES, B. C., ZANGVILL W. I.: *Generalized Cutting Plane Algorithms.* SIAM J. Control, Vol. 9, No. 4, 1971.
7. VEINOTT A. E.: *The Supporting Hyperplane Method for Unimodal Programming.* SIAM J. Control, 15, 1967.
8. TOPKIS D. M.: *Cutting Plane Methods without Constraint Sets,* O.R., 18, 1970.
9. SPIRCU L.: *Unele aspecte privind metodele de rezolvare a problemelor de programare neliniară.* (Some Aspects Concerning Methods for Solving Non-linear Programming Problems). Studii și cercetări de calcul economic și cibernetică economică, No. 1, 1974.
10. FABIAN CS.: *Some Algorithms for Non-linear Integer Programming.* Economic Computation and Economic Cybernetics Studies and Research, No. 3, 1976.
11. MARUSCIAC I.: *Metode de rezolvare a problemelor de programare neliniară* (Methods for Solving Non-linear Progarmming Problems), Ed. Dacia, Cluj, Romania, 1973.
12. LAWLER E. L., BELL M. D.: *A Method for Solving Discrete Optimization Problems.* O. R., Vol. 14, 1967.

ETUDE COMPARATIVE DE CERTAINES METHODES DU TYPE "PLAN DE SECTION" DANS LA PROGRAMMATION MATHEMATIQUE NONLINEAIRE CONVEXE

L'ouvrage présente les résultats comparatifs de certaines méthodes du type "plan de section" pour quelques problèmes — tests de programmation nonlinéaire connus en littérature.

Furent testées les méthodes de Kelley [1], Kleibohm [2], Zoutendijk [3], Wolfe [4], ainsi qu'une variante modifiée de la méthode de Kelley.

СРАВНИТТЕЛЬНОЕ ИЗУЧЕНИЕ НЕКОТОРЫХ МЕТОДОВ ТИПА «CUTTING PLANE» В НЕЛИНЕЙНОМ ВЫПУКЛОМ МАТЕМАТИЧЕСКОМ ПРОГРАММИРОВАНИИ

В работе дются сравнительные результаты некоторых методов типа «Cutting Plane» для нескольких, известных в литературе, задач-тестов нелинейного программирования.

Был проверени метод Келлей (1), Клайбон (2), Зутенжик 3), Вольфе (4), а также измененный вариант метода Келлей.

20.II.1977