G. DODESCU
Department of Economic Cybernetics
Academy of Economic Studies

Consistency and Stability of a Discrete Analog of State Estimator

An important problem of systems theory is the building of the regulator with a state estimator. In this paper, we consider a case directly reflecting the reality, namely the achievement of the regulator with the state estimator on a digital computer directly connected to the given system.

We consider a Σ linear and constant system in the distributions space $\mathcal{D}_+$ with the support on the real and positive semiaxis R_+:

$$\delta'_* x = Ax + Bu \qquad (\Sigma)$$
$$y = Cx$$

where A, B, C, are real matrices and x is the state:

$$x = \text{col}(x_1\ x_2 \dots x_n); \qquad x_i \in \mathcal{D}'_+; \qquad i = 1(1)\ n.$$

System input is u:

$$u = \text{col}(u_1\ u_2 \dots u_m); \qquad u_i \in \mathcal{D}'_+; \qquad i = 1(1)m.$$

System output is y:

$$y = \text{col}(y_1\ y_2 \dots y_p);\ \text{it follows}\ y_i \in \mathcal{D}'_+; \qquad i = 1(1)\ p.$$

Now, we consider a state estimator $\hat{\Sigma}$ of system Σ as follows:

$$\delta'_* \hat{x} = A\hat{x} + L(y - C\hat{x}) + Bu$$
$$\hat{y} = \hat{x}$$

and the control law:

$$u = -K\hat{x}.$$

It is assumed to occur a disturbance at $t = 0$, under the form:

$$x_0\delta \rightarrow \text{for system } \Sigma$$

and

$$\hat{x}_0\delta \rightarrow \text{for system } \hat{\Sigma}$$

where x_0 and $\hat{x}_0$ can take n constant and real values, and δ is Dirac's distribution having the support in $t = 0$.

We mention that if (A, B) is entirely controllable, then for any state disturbance under the above form, it exists u which brings the system in zero. A similar mode of treating would be therefore the introduction of a disturbance $u_0\delta$ at system input.

The solution for the equation of state estimator is:

$$\hat{x} = e^{(A-LC)t} * (Ly + Bu). \tag{1}$$

We shall compute the system response formed of system Σ, state estimator $\hat{\Sigma}$ and control connection $u = -K\hat{x}$.

To this end, taking into account the state disturbances, we have:

$$\delta'_* x = Ax - BK\hat{x} + x_0\delta$$

$$\delta'_* \hat{x} = A\hat{x} - BK\hat{x} + LC(x - \hat{x}) + \hat{x}_0\delta.$$

If we denote by

$$\tilde{x} = x - \hat{x}$$

and we subtract the two equations, one gets:

$$\delta'_* \tilde{x} = (A - LC)\tilde{x} + \tilde{x}_0\delta$$

after ordering

$$(\delta' I - A\delta + LC\delta) * \tilde{x} = \tilde{x}_0\delta$$

and the solution is:

$$\tilde{x} = e^{(A-LC)t}\tilde{x}_0. \tag{2}$$

This solution is got directly in $\mathcal{D}'_+$, because the inverse of the matrix $(\delta' I - \delta A + \delta LC)$ exists and follows through "dividing" on the direction of convolution inverse.

$$\frac{\delta I}{\delta' I - \delta A + \delta LC} = I\delta^{-1} + (A - LC)\,\delta^{-2} + (A - LC)^2\delta^{-3} + \ldots$$

On the other hand we have:

$\delta^{-1} = h(t)$ Heaviside's distribution

$\delta^{-2} = t_+$

$\delta^{-3} = \frac{t_+^2}{2!}$

a.s.o.

The number is convergent for $t \geqslant 0$ and for any matrix $A - LC$, therefore the solution is (2). Because we have:

$$\hat{x} = x - \tilde{x}$$

it follows:

$$\delta'_* x = (A - BK)\, x + BK\tilde{x} + x_0\delta$$

and further on:

$$(\delta' I - \delta A + \delta BK) * x = BKe^{(A-LC)t}\tilde{x}_0 + x_0\delta$$

hence the solution follows:

$$x = e^{(A-BK)t} * (x_0\delta + BKe^{(A-LC)t}\tilde{x}_0). \tag{3}$$

The solution under the form (3) offers us an interesting discussion: if system Σ and system $\widehat{\Sigma}$ are subjected to the same disturbance, then we have: $x_0 = \hat{x}_0$ and therefore $\tilde{x}_0 = 0$.
Thus, it results in this case that:

$$x = e^{(A-BK)t}\, x_0 \tag{4}$$

this meaning that the controlled system behaves as if the control would be achieved directly after the state of Σ, namely with the control law got without a state estimator, under the conditions of the state availability of Σ.
But, because generally the disturbances are not identical for Σ and $\widehat{\Sigma}$, the response for the disturbances is (3), which includes (4), but one adds the effect of disturbances inequality:

$$(\tilde{x}_0 \neq 0).$$

After this general character introduction, let's analyse the case where the state estimator is a discrete analog of $\widehat{\Sigma}$ as follows: we consider the estimator input as being sampled, with the sampling interval T.

$$u = T\sum_{k=0}^{\infty} u_{KT}\delta_{KT}; \qquad K \in N$$

where u_{KT} = constant and δ_{KT} is a Dirac's distribution with the starting point in KT. We approximate the deviate through the difference:

$$\delta' \cong \frac{\delta_{-T} - \delta}{T}$$

and follows:

$$\frac{\delta_{-T} - \delta}{T} * \hat{x} = A\hat{x} + L(y - C\hat{x}) + BT\sum_0^\infty u_{KT}\,\delta_{KT}.$$

In this equation y is considered sampled (synchronous) with the same T period. The solution of this equation is ascertained by means of the same method, namely:

$$\left(\frac{\delta_{-T} - \delta}{T} I - A\delta + LC\delta\right) * \hat{x} = Ly + BT\sum_{k=0}^\infty u_{KT}\,\delta_{KT} = L\sum_0^\infty y_{KT}\,\delta_{KT} +$$

$$+ BT\sum_0^\infty u_{KT}\,\delta_{KT}.$$

The inverse of the matrix is computed through "dividing"; we denote by:

$$I + T(A - LC) = \mathcal{A}$$

and we have:

$$T(\delta_{-T}I - \delta\mathcal{A})^{-1} = T(I\delta_T + \mathcal{A}\delta_{2T} + \mathcal{A}^2\,\delta_{3T} + \ldots) = T\sum_0^\infty \mathcal{A}^k\,\delta_{(k+1)T}$$

and the solution is:

$$\hat{x} = \left(T\sum_0^\infty \mathcal{A}^k\,\delta_{(k+1)T}\right) * \left(LT\sum_0^\infty y_{KT}\,\delta_{KT} + BT\sum_0^\infty u_{KT}\delta_{KT}\right). \qquad (5)$$

The consistency problem is studied as follows: if the sampling step tends towards zero, then (5) has to tend towards (1). To prove this we use a judgement of passing to the limit "term by term" (that is not rigurous, but the result is true and the method is simple). Thus, we prove that (under the conditions of $\varphi \in \mathcal{D}$ with a compact):

$$\lim_{\substack{T\to 0\\ k\to\infty}} < T\sum_0^\infty (I + T(A - LC))^k \delta_{(k+1)T}\varphi(t) > = < e^{(A-LC)t}, \varphi(t) >$$

we have indeed

$$\lim_{\substack{T\to 0\\ k\to 0}}\left(\sum_0^\infty T(I + T(A - LC)^K < (t - (K+1)\,T), \quad \varphi(t) > \right) =$$

$$\lim_{\substack{T\to 0\\ k\to\infty}}\left(T\sum_0^\infty (I + T(A - LC))^K \quad \varphi((K+1)\,T) = \int_{R_+} e^{(A-LC)t}\varphi(t).\right.$$

As regards the terms in the second bracket, the same mode leads to:

$$\lim_{\substack{T\to 0\\ k\to\infty}} < LT\sum_0^\infty y_{KT}\,\delta_{KT}, \quad \varphi(t) > = L\int_{R_+} y(t)\,\varphi(t).$$

Here, we assumed implicitly that $y(t)$ is a continuous translation function, namely $y(a)$ exists and is unique for any $a \in R$. Similarly:

$$\lim_{T \to 0} < TB \sum_{0}^{\infty} u_{KT} \delta_{KT} \quad \varphi(t) > = B \int_{R_+} u(t)\, \varphi(t)$$

under the same conditions of continuous translation for $u(t)$. The hypothesis of continuous translation is necessary, because otherwise the signal can not be sampled.

It follows from the above problems that the discreet is consistent in x, in u and in y, therefore the discrete $\widehat{\Sigma}$ is consistent with $\widehat{\Sigma}$.

We may define the total error of discreet as follows: we denote by $\hat{x}$ solution (1) and by $\hat{x}'$ solution (5). Then, the total error of
is:

$$\varepsilon = \max_{k} | \hat{x}_{KT} - \hat{x}'_{KT} | .$$

We mean by stability that one of a dynamic system; thus, $\widehat{\Sigma}$ is constant if all the proper values of matrix $A - LC$ have the real side negative. The discrete system $\widehat{\Sigma}$ is considered analogically as being constant and thus, the matrix $\mathcal{A} = I + T(A - LC)$ must has a subunitary spectral radius:

$$\delta(\mathcal{A}) < 1.$$

Denoting by λ_i the proper values of matrix $\mathcal{A}$, this condition goes back to

$$\max_{i} | \lambda_i | < 1.$$

Therefore, if $\rho(I + T(A - LC)) < 1$ then, all the proper values of matrix $T(A - LC)$ are placed into a disk with the centre in point -1 and with the subunitary radius. Thus, all the proper values of matrix $T(A - LC)$ have the real side negative.

Because $T > 0$, it follows finally that also matrix $A - LC$ has all the proper values with the real side negative.

Conversely, we assume that all the values of $A - LC$ have the real side negative, because they are finite, they are all placed into a bounded field of $-a \pm ib$ and $-(a + b) \pm ib$ and the proper values of matrix $I + T(A - LC)$ are placed into a bounded field of $I + T(-a \pm ib)$ and $1 + h(-a - b \pm ib)$. Consequently, for the condition $\rho(I + T(A - LC)) < 1$ it is enough that:

$$(1 - aT)^2 + T^2 b^2 < 1$$

$$[1 - (a + b)T]^2 + T^2 b^2 < 1.$$

These conditions are both met for a small enough T. Finally, if $\widehat{\Sigma}$ is constant, then also the discrete $\widehat{\Sigma}$ is constant for a small enough T.

So far, we assumed that the stability is defined as an "asymptotic stability"; it is not difficult to extend the above judgement also for the stability case on a larger scale (Liapunov).

Using Jordan canonical form, we prove that for $\widehat{\Sigma}$ the stability condition becomes in this case:

$$\rho(I + T(A - LC) \leqslant 1.$$

The control analysis with a discrete state estimator is generally a complicated problem. Referring to the basic diagram in Fig. 1, the equations describing this total system are got as follows: the sampling body is considered as being described by input u and output $UT\sum_{0}^{\infty}\delta_{KT}$ where u is a continuous translation function. The shift lock (the transition from the sampled signal to the continuous one) is described by:

$$u_b(t) = \left(UT\sum_{0}^{\infty}\delta_{KT}\right) * \alpha(t)$$

where

$$\alpha(t) = \begin{cases} 1 \text{ for } t\in[0.1] \\ 0 \text{ for } t\in[0.1]. \end{cases}$$

It is noticed that $u_b(t) \neq u(t)$, namely through sampling and locking, the initial signal does not result, using an adequate filtration, the locked signal can be rather closed to the initial signal.

Increasing the computing speed of digital units allows the introduction of certain stimulators using this kind of units, getting the advantages specific to this technique for real time working schemes.

For instance, using the terms of "frequency" to some control schemes in certain fields "the computation" is carried out at a frequency of 200 hz, the connection between the continuous side and the discrete one being assumed by analog to digital converters and conversely. More, the computation unit called "adaptive control" appears at one of these schemes and allows the modification of the matrices A, B, C, depending on the requirements of the model through an identification block. Here, we meet the computing speed too as an essential element conditioning the achievement. The main idea is the simulation of system Σ with a variable structure by a simulator and an identification block following the structure variation assured by a sufficient number of recording devices supplying the supplementary and necessary information.

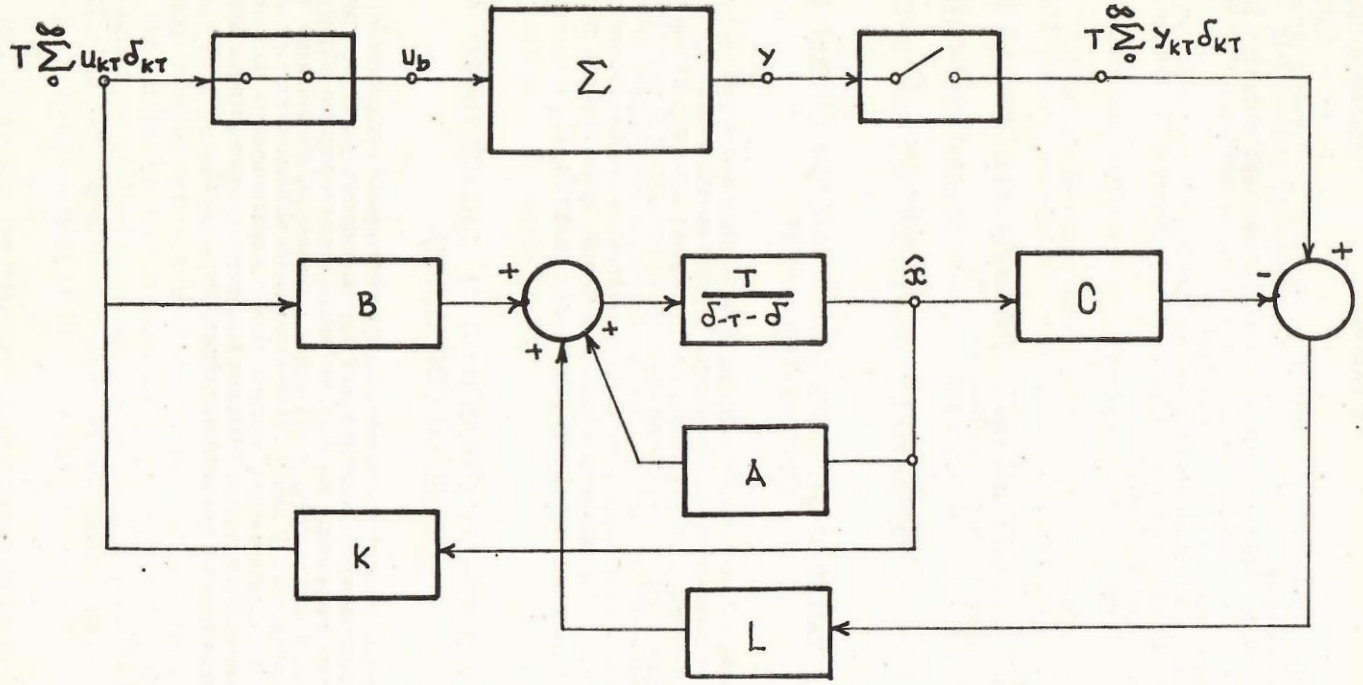
$T\sum_0^\infty u_{kT}\delta_{kT}$
u_b
Σ
y
$T\sum_0^\infty y_{kT}\delta_{kT}$
B
+
+
+
$\frac{T}{\delta - \tau - \delta}$
$\hat{x}$
C
-
+
A
K
L

Fig. 1

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CONSISTANCE ET STABILITE D'UN ANALOGUE DISCRET DE L'ESTIMATEUR D'ETAT

On considère un système linéaire différentiel et constant couplé à un calculateur digital à l'aide duquel on fait la simulation du système en obtenant un estimateur asymptothique d'état. On prend en considération que le calculateur opère avec des signaux — échantillon. Dans ces conditions apparaît le problème de la consistance, de la convergeance et de la stabilité du système entier. Dans le présent article on prend en considération aussi, le cas d'un système adaptatif ayant un bloc d'identification pouvant être approximé sur les intervalles par une séquence de systèmes constants. Le traitement, à l'aide de la théorie des distributions, permet la simplification des operations de calcul dans l'algèbre de convolution $\mathcal{D}'_+$.

УСТОЙЧИВОСТЬ И СТАБИЛЬНОСТЬ ДИСКРЕТНОГО АНАЛОГА ОЦЕНКИ СОСТОЯНИЯ

Рассматривается линейная система Σ дифференциальная и посьоянная, соединенная с вычислительной машиной, в которой имитируется система, реализуя асимптотическую оценку состояния. Учитывается тот факт, что вычислительная машина оперирует выборочными сигналами. В этих условиях встает вопрос устойчивости, конвергентности и стабильности глобальной системы. В данной статье принимается во внимание и случай адаптивной системы с блоком идентификации, которую можно апроксимировать по интервалам при помощи серии постоянных систем. Трактовка посредством теории распределения позволяет упрошение вычислительных операций в алгебре конволиции D'_+.

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