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Qualitative Models in Economic Processes

The paper presents a new modelling method of dynamic processes, as an application of catastrophe theory.

Making use of some notions of the theory of Thom, a new (qualitative) modelling method is presented, starting with a survey of the classical (quantitative) method.

This new technique is illustrated with an example.

Although the theory has a very difficult mathematical background, only its intuitive and practical aspects are pointed out.

1. SURVEY OF THE CLASSICAL MODELLING METHOD [1].

By studying the laws governing dynamic processes, one may write their model, represented by a set of equations of the form:

$$\frac{dx_i}{dt} = X_i(x_j, \tau, t), \quad \text{where} \tag{1}$$

x_j are the states of the system (process), $j = 1, \ldots n$,
τ is an external parameter
t denotes the time
$M = R^{n+2}$ is the phase of the system (1) made of points (x_j, τ, t).
X_i: $M \to R_,$, $i = 1, \ldots, n$, define a vector field X, on M.

Under some regularity conditions, such as differentiability, one can integrate the system (1), obtaining the equations (2)

$$x_i = h_i(x_i^\circ, \tau, t)$$

(the integration is local, in a neighborhood of the initial values $(x_i^\circ, \tau^\circ, t^\circ)$.

This technique can be employed when the laws which govern (locally) the process can be identified.

The most remarkable example was given by Newton, in Celestial Mechanics; by means of his Gravitation Law $\left(F = \frac{GmM'}{r^2}\right)$ defining X, it became possible to determine precisely the orbits of the planets, solving a system similar to (1).

This success was the main reason of the introduction of quantitative modelling in science, an approach which aims to determine precisely the functions h_i.

Unfortunately, there are some difficulties when we want to use this kind of modelling:

— There are few natural phenomena (and much less economic and social ones) which allow an exact mathematical description. In many cases X is given by approximate formulae.

— Even if we know X, in very few cases, we are able to integrate the system (1), so we obtain only approximated solutions.

It appears that the programme of quantitative dynamics (and modelling) is, in many cases, too ambitious. A "weakening" of this programme, but still useful, is due to H. Poincaré, the pioneer of qualitative dynamics.

We note that now, the expected answers are qualitative rather than quantitative: will the collapse hold or not? and we are not so interested in the magnitude of the efforts in every point of the lids. So the interest moves from the quantitative determination of the solutions, towards the global, geometric study of the solutions, towards their assimptotic behaviour.

A new step forward was done, when Andronov and Pontrjagin introduced the notion of structural stability.

The problem of structural stability derives from the practical need of observing the phenomena. (A phenomenon can be studied only by reapeated observations, but if it is not structurally stable, a slight perturbation can change completely its structure and properties, hence, practically, it cannot be studied).

Intuitively, we may see structural stability as the property of a system which conserves (in a certain sense) its structure, when small perturbations occur in its parametra.

For the first time, this notion was applied to dynamical systems, (the system $\dot{x} = ax$, $a > 0$ is structurally stable, while $\dot{x} = ax$, $a = 0$ is not).

The question which naturally arises is to know if the set of structurally stable systems (defined on M) is dense in the set of dynamical systems (on M). That means that any dynamical system, can be, to any desired extent, approximated by a structurally stable system. (For an interesting discussion on this matter see also [2]).

This problem was solved by Peixoto who showed that the density is valid for dim $M \leqslant 2$, but Smale gave a counterexample which shows that for dim $M \geqslant 4$ this property is not true.

One can see that the notion of structural stability applied to dynamical systems is too "strong" for an effective employment.

Thom ([3]), suggested a "weaker" application of the structural stability: not to dynamical systems (which implies taking wholly into account all the trajectories) but to the final states of the system (the "end" of the trajectories), in fact to the attractors of the system (which may be considered as "stable equilibria" of the system).

This seems a natural suggestion, since in many cases we are more interested in final states, than in the evolution of the system untill the vicinity of these states.

The question that arises (in the sense of structural stability) is, if the set of dynamical systems (in fact the vector fields X) with structural unstable attractors could be locally dense in the set of dynamical systems.

(One should note that instead of studying the density of the systems with structural stable attractors, one tries to solve the "opposite" problem, that of the local density of the bifurcation set K, hence of nongeneric points, which represent structurally nonstable systems).

The general answer to this problem is still unknown. In order to find an answer, a useful step is a classification of the generic forms of bifurcation.

(The bifurcation set having a certain form, we are interested only in those K sets which are structurally stable, that is which preserve in a certain sense their form, when slight perturbation occur).

Remarkable results have been obtained in the case of gradient systems. ($X = -$ grad V, $V =$ potential function).

Now, the attractors of X are the local minima of V. Thus, the problem shifts towards the study of the singularities of V (points where all the partial derivatives $\frac{\partial V}{\partial x_j}$, vanish).

For gradient systems of condimension $\leqslant 4$ (the codimension of a singular point is the number of topological types according to which the function deformes around this point), thanks to Mather's works on the singularities of differentiable mappings, a finite classification of the generic forms of the bifurcation set was obtained.

When the codimension is not greater than 4, seven types of forms were obtained, the so called elementary catastrophes, if they are interpreted as dynamic fields (see also [4]).

2. THE MODELLING METHOD.

This method comes from Thom's theorem, the main result of catastrophe theory.

Let J, the space of C^∞ functions defined on R^{r+k} (n states, k control parametra) with the Whitney topology. (They are in fact potential functions).

According to the theorem, if I_0 is the set of all the functions f, which pertain to the seven classes, J_0 is an open and dense subset of J; the function of these classes are "structurally stable" in the sense that a small perturbation on f does not change its "catastrophic" behaviour.

Thus, concerning the modelling, one can say that for gradient systems, depending upon at most 4 control parametra, any system (in fact any potential function) can be to any desired extent, approximated by a potential function, belonging to one of the above-mentioned seven classes. (For examples see [5], [6], [7]).

We add that the finiteness of the classification holds for $k = 5$, too, but no further. We have to say also that in the classification, k is the essential parameter, but for these seven classes $n = 2$.

The modelling technique can be summarized as follows:

— Check if the system attempts to minimize locally a function f (it is a gradient system).

— In this case the potential function will almost sure belong to one of the seven mentioned classes (because J_0 is dense and open in J).

— Assign the function to one or other of the classes by testing the process (studying its local minima), for different sets of parametra. (The strategy of choosing these sets had not been studied yet). These local tests are in a sense similar to a quantitative modelling.

If the experimental results concur with those known from one of the seven classes, we can assign the function to that class.

— The precise determination of the function within a certain class can be done as follows: we established the form of the model; the unknown coefficients can be determinated if we dispose of sufficient sets of experimental data, by means — for instance — of the least squares method.

— This kind of modelling is "robust" (within J_0), to perturbations (in the sense of structural stability), according to the theorem.

3. EXAMPLE: FUND ALLOCATION FOR RESEARCH AND DEVELOPMENT

The problem of determining the optimal ratio between the supply needs and the required number of research workers, will be studied below.

The best (optimal) way of assigning the available funds between these two kinds of expenses will be established at the same time.

Once the model established, one can employ it in forecasting too.

We follow the technique presented in 2.

Since a local optimum is to be reached, we may consider to be in the class of gradient systems.

We suppose that the value of technical means exceeds the remuneration of research workers, because "technological" research has priority as compared to other fields.

We also make the assumption that the period of time is the year.

Let:

$K_e(t)$ = the existing funds embodied in technical means at moment t.

$k_{in}(t)$ = the funds that will be invested during the next period of time.

$L_e(t)$ = the number of research workers already employed at moment t.

$L_{in}(t)$ = the number of research workers that will be employed during the next period of time.

It should be noted that

$$\begin{aligned} K_e(t+1) &= K_e(t) + K_{in}(t) \\ L_e(t+1) &= L_e(t) + L_{in}(t). \end{aligned} \tag{3}$$

Now, we shall introduce the defining elements of the model:

M is the state space of the process (the model of which is to be established).

$m \in M$, represents the ratio between the funds already accumulated in technical equipment and the remuneration of the employed research workers.

We may consider $M = R$ and

$$m(t) = \frac{K_e(t)}{12\, aL_e(t)}, \text{ where} \tag{4}$$

a = the monthly average retribution for a research worker

$E: M \to R$ is the profit function, according to which the optimum of fund distribution will be established (optimum = maximum of efficiency).

(Our aim is to determine the expression of the profit function).

This function also depends upon the parametra of a space U, that is $E: U \times M \to R$. Then, the potential function is $V = -E$.

The two parametra that will be considerd are:

$X_1 = F_e(t)$ funds already existing in research, in the present year, $t(F_e(t) = K_e(t) + 12\ a\, L_e(t)$.

$X_2 = F_{in}(t)$ = funds that will be spent for research, the next year $(F_{in}(t) = K_{in}(t) + 12\ a\ (L_e(t) + L_{in}(t)$.

We consider $U = R^2$.

Now, we shall study the "qualitative" behaviour of the optimal ratios, m^*.

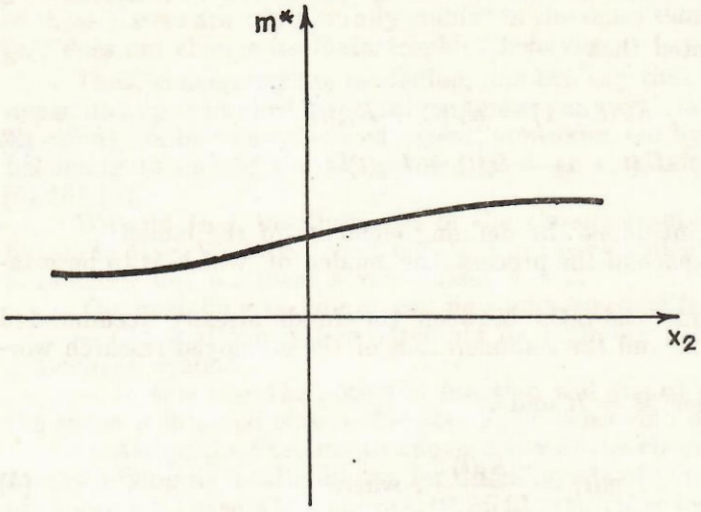
For a "small" x_1, that means not a verry developed research, we may consider only one possibility of an optimal allocation of investment. This seems to be natural, since the only possibility consists in the development of the material endowment of the research.

For a big x_1 — that means for a developed research, we may consider two posibilities of optimal allocation of investments. This seems also natural, the two possibilities being in accordance with two, generally diferent, local optima:

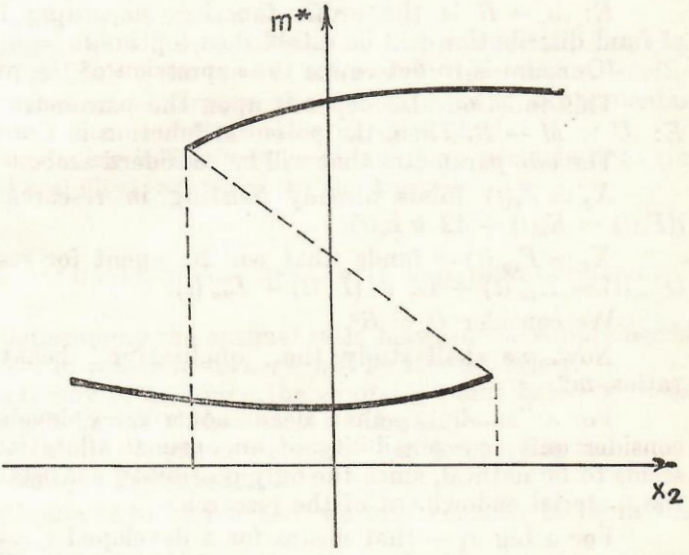
— The smallest, an investigating stretegy of "extensive" type (the ratio equipment/research workers is smaller).

— The highest, represents an "intensive" investigating strategy (the ratio is higher).

All these are illustrated in the diagrams below:



x_1 = "small"



x_1 = "high".

Then, a global dynamic system, which would present a phenomenon like that above, will have as potential function (or family of potential functions $V = -E(m, x_1, x_2)$, with the following characteristics: for a region of the parameter space, V has a single minimum and there is a region of the parameter space where V has two local minima (the common frontier of the two regions will be the set of catastrophes).

Because dim $M = 1$, the proper mathematical model will be that indicated by the Riemann — Hugoniot catastrophe (according to Thom's theorem), having the potential function:

$$V = \frac{m^4}{4} - \frac{m^2}{2} x_1 - m x_2 \quad (M = R,\ U = R^2).$$

We established in this way the class to which our model belongs.

In order to ascertain it, we shall write it as bellow:

$$E(m, x_1, x_2) = a\, m^4 + b m^2 x_1 + c m x_2.$$

If we know a series of values (E, m, x_1, x_2) the coefficients a, b, c can be determined by means of the least squares method.

This model is "reliable", because E, according to Thom's theorem is "structurally stable"; as a consequence, slight variations in value of the coefficients does not affect it essentialy.

The error function is $J(a, b, c) = \sum_{i=1}^{n} (E_i - (a m_i^4 + b m_i^2 x_1^i + c m_i y_1^i$. (Here i is an index denoting the time). From the minimum error condition, $\frac{\partial J}{\partial a} = 0$, $\frac{\partial J}{\partial b} = 0$, $\frac{\partial J}{\partial c} = 0$, we obtain the system of equations:

$$D \begin{bmatrix} a \\ b \\ c \end{bmatrix} = f, \text{ where}$$

$$d_{11} = \sum_{i=1}^{n} m_i^8, \quad d_{12} = \sum_{i=1}^{n} m_i^6 x_1^i, \quad d_{13} = \sum_{i=1}^{n} m_i^5 x_2^i$$

$$d_{21} = \sum_{i=1}^{n} m_i^6 x_1^i, \quad d_{22} = \sum_{i=1}^{n} m_i^4 (x_1^i)^2, \quad d_{23} = \sum_{i=1}^{n} m_i^3 x_1^i x_2^i$$

$$d_{31} = \sum_{i=1}^{n} m_i^5 x_2^i, \quad d_{32} = \sum_{i=1}^{n} m_i^3 x_1^i x_2^i, \quad d_{33} = \sum_{i=1}^{n} m_i^2 (x_2^i)^2$$

$$f_1 = \sum_{i=1}^{n} m_i^4 E_i, \quad f_2 = \sum_{i=1}^{n} m_i^2 x_1^i E_i, \quad f_3 = \sum_{i_i=2}^{n} m_i x_2^i E_i$$

and (E_i, m_i, x_1^i, x_2^i), $i = 1, \ldots, n$ are the sets of data which must be known (or determined) from the beginning. Here, the model is established via its coefficients, by the formula

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = D^{-1} f.$$

Concerning the data, we note that
n must be at least 3

$$x_1^i = F_e(i),\ X_2^i = F_{in}(i),\ m_i = \frac{K_e(i)}{12\,aL_e(i)}.$$

For the efficiency estimation we adopted (as a first step),

the formula $E^i = \frac{\alpha V'(i)}{F_{in}(i)}$, where α is the rate of technical progress, evaluated from a Cobb-Douglas type production function. Then, E_i represents the ratio between the share of G.N.P. due to the technical progress and the investments in scientific research.

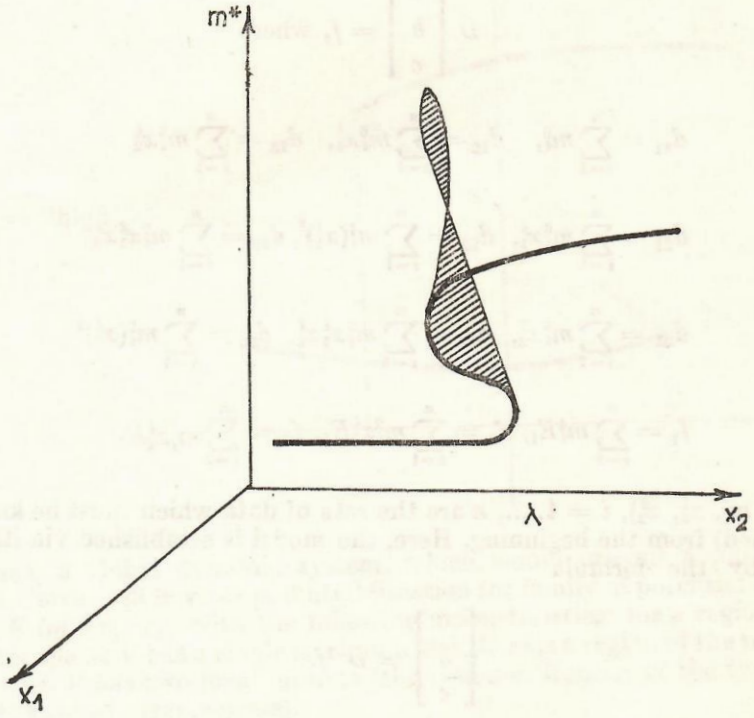
The set of data is now determined it we know
$F_e(i)$, $F_{in}(i)$, $K_e(i)$, $L_e(i)$, a, $V'(i)$, which may be found in statistics.

For the data used, we obtained a model (the expression of E), which was verified and yielded a 10% error.

Once the model established, it is known that the optimal ratios (for the research funds) m^*, are given by the maxima of E (or the minima of V), as a function of the parametra x_1, and x_2.

These lie on the upper and lower sheet of a surface (of the Riemann — Hugoniot type, as in figure), obtained from the (maximum) condition $\frac{\partial E}{\partial m} = 0$.

(the axes are shifted)



Hence, if at moment t the process is determined by the parametra $x_1(t)$, $x_2(t)$, then on the "vertical of the point", we find the corresponding optimal $m^*(t+1)$ (if there are two values, we choose one, according to some aditional criteria).

Then, if we know $K_e(t)$ and $L_e(t)$, making use of (4) (for the moment $t+1$) and (3) (for the moment t), we obtain the optimal allocation of $F_{in}(t)$ between $K_{in}(t)$ and $L_{in}(t)$.

The same model can be employed in forecasting. This is because the model is essentially global, in the sense that the evolution of the phenomenon is seen as explicitly depending upon several parametra (in our situation x_1 and x_2). If the model would be local, a small change in the parametra would make it (theoretically) unsuitable for projections.

In our case, projection of K_e an L_e reduces to projection of x_1 and x_2 (for a remote t), applying the upper method.

4. FINAL REMARKS.

Although the mathematical tool of the catastrophe theory is very evoluated, the modelling technique is quite accessible.

The advantages of this modelling method seem to be:

— The model is established by "choosing" it from a finite number of possibilities (7), instead of an "elaborated" local study.

— The model is global (by particularisation one obtains local models).

— The possibility of optimisation arises naturally if the potential function is an objective function.

The application of this method is however, restricted to gradient systems.

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MODELES QUALITATIFS DANS LES PROCESSUS ECONOMIQUES

Les auteurs présentent une méthode nouvelle (qualitative) de mise en modèle d'une certaine catégorie de processus dynamiques, prenant comme point de départ quelques concepts de la théorie des catastrophes élaborée par Thom, qui font la critique de la méthode classique (quantitative). La nouvelle méthode de mise en modèle est illustrée d'un exemple.

Bien que cette méthode ait pour fondement une théorie mathématique qui présente des difficultés, on met l'accent sur son coté intuitif, applicatif.

КАЧЕСТВЕННЫЕ МОДЕЛИ В ЭКОНОМИЧЕСКИХ ПРОЦЕССАХ

В статье представлен новый метод моделирования одной категории динамических процессов, исходя из теории катастроф.

Используя некоторые понятия теории, выработанной Тхомом, автор показывает новый метод моделирования (качественного), исходя из критики классического метода (количественного). Новый метод моделирования проиллюстрирован одним примером.

Хотя в основе этого метода лежит очень сложная математическая теория, его интуитивная, практическая сторона менее сложна, и как раз она акцентируется в статье.

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