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Linear Fractional Functional Programming with a Parameter in an Activity Vector

This paper presents an algorithm that will parametrize an activity vector for a linear fractional functional programming problem. Firstly the problem has been solved by a method keeping the parameter Φ at zero level. Then, a search for a sequence of values of the parameter Φ viz $\Phi^1, \Phi^2, ..., \Phi^k$ and a series of corresponding basis $B^1, B^2 ... B^k$ is carried out in such a way that B^k is optimum feasible for all $\Phi^T \leqslant \Phi \leqslant \gamma$ or unbounded or infeasible for $0 \leqslant \Phi - \Phi^T \leqslant \in$ where $\in$ is an arbitrary small positive number and γ is, any real number.

The present paper considers a case in which only one activity vector is to be parametrized by simple transformations, provided the problem under consideration is stable and there is no disconnection in the feasible space for the parameter.

The algorithm may provide solutions to a fairly large number of problems because there may be a class of problems for which the instability and disconnection of the feasible space may never occur. In case a disconnection occurs, one should not pursue the parametrization process any further since any disconnection implies a fundamental structural change between the two feasible regions, therefore one has to reassess the entire problem rather than merely extending the parametrization process. Generally the disconnection does not occur at least within the range of variation of the parameter because of economic realities. The problem and definitions are introduced in the first section of the paper, where as the feasible region for the parameter Φ (given an initial optimal basis) and successive transformations to a new basis are discussed in the last section of the paper.

The parametric linear fractional functional programming problem to be solved here is that of finding a sequence of optimum feasible solutions to

$$\text{Maximize } \frac{Z_1}{Z_2} = \frac{\sum_{i=1}^{n} C_i X_i + \alpha}{\sum_{i=1}^{n} d_i X_i + \beta}.$$

$$\text{Subject to } \sum_{j=1}^{n} = X_j A_j + X_p(\bar{A}_p + \Phi \hat{A}_p) = b \tag{1.1}$$

$$j \neq p \quad X_j \geqslant 0 \quad j = 1, 2 \ldots . n$$

where

(1) A_j and b are $m \times 1$ vectors.

(2) Φ, γ, C_j and d_j are real numbers.

It is assumed that

(a) $\sum_{j=1}^{n} (d_j X_j + \beta) \neq 0$ over the feasible region which is regular (bounded and nonempty).

b) γ is given for the problem and satisfies $0 \leqslant \gamma < \infty$.

(c) $A_p = \bar{A}_p + \Phi \hat{A}_p$.

(d) optimal basis exists for $\Phi = 0$.

Let $B^k(\Phi)$ denote the basis matrix for a given value of Φ and let

$$X_B^K(\Phi) = [B^k(\Phi)]^{-1} b$$

$$Y_j^k(\Phi) = [B^k(\Phi)]^{-1} A_j$$

$$Z_{1j}^k(\Phi) = C_B^k [B^k(\Phi)]^{-1} A_j \tag{1.2}$$

$$Z_{2j}^k(\Phi) = d_B^k [B^k(\Phi)]^{-1} A_j$$

$$t_j^k(\Phi) = Z_{2B}^k(\Phi)[Z_{1j}^k(\Phi) - C_j] - Z_{1B}^k(\Phi)[Z_{2j}^k(\Phi) - d_j]$$

where C_B^k and d_k^B contain m basis prices, $Z_{1B}^k(\Phi)$, $Z_{2B}^k(\Phi)$ denote the values of Z_1 and Z_2 at the optimum basis $B^k(\Phi)$.

The superscript k indicates the kth basis among the sequence of basis obtained through the parametrization process. This is used just to distinguish two equally optimum feasible bases $B^k(\Phi)$ and $B^T(\Phi)$ for a given value of the parameter Φ. The suffix $k = 1$ denotes the initial optimum basis.

Denote

$$y_p^k(\Phi) = \bar{y}_p^k(\Phi) + \Phi\hat{y}_p^k(\Phi)$$

$$Z_{1p}^k(\Phi) = \bar{Z}_{1p}^k(\Phi) + \Phi\hat{Z}_{1p}^k(\Phi)$$

$$Z_{2p}^k(\Phi) = \bar{Z}_{2p}^k(\Phi) + \Phi\hat{Z}_{2p}^k(\Phi)$$

where

$$\bar{y}_p^k(\Phi) = [B^k(\Phi)^{-1}\bar{A}_p \qquad \hat{y}^k(\Phi) = [B^k(\Phi)]^{-1}\bar{A}_p$$

$$\bar{Z}_{1p}^k(\Phi) = C_B^k[B^k(\Phi)]^{-1}\bar{A}_p \qquad \hat{Z}_{1p}^k(\Phi) = C_B^k[B^k(\Phi)]^{-1}\hat{A}_p$$

$$t_p^k(\Phi) = Z_{2B}^k(\Phi)[Z_{1p}^k(\Phi) - C_p] - Z_{1B}^k(\Phi)[Z_{2p}^k(\Phi) - d_p].$$

Search for Feasible region for Φ for an initial optimal basis.

Assume that an optimum feasible basis has been obtained for $\Phi = 0$. Now let $\Phi = \Phi' + \delta$ where $\Phi' = 0$ and $\delta \geqslant 0$; then $B^1(\Phi) = B^1(\delta)$ and the corresponding values of $X_B^1(\Phi)$, $Y_j^1(\Phi)$, $Z_{1j}^1(\Phi)$, $Z_{2j}^1(\Phi)$, $t_j^1(\Phi)$ are obtained by replacing $\Phi = \delta$ in (1.2). Thus $B'(\Phi)$ is optimum feasible for $0 \leqslant \Phi \leqslant \theta$; then $B^1(\delta)$ is also optimum feasible for $0 \leqslant \delta \leqslant \theta$.

Moreover $t_p^1(\Phi' + \delta) = \{[Z_{2B}^1(\Phi' + \delta)\,[\bar{Z}_{1p}^1(\Phi') + \delta\hat{Z}_{1p}^1(\Phi') - C_p]$

$$- Z_{1B}^1(\Phi' + \delta)\,[\bar{Z}^1{}_p(\Phi') + \delta\,\hat{Z}_{2p}^1(\Phi') - d_p]\} \geqslant 0$$

for the optimal basis $B^1(\Phi' + \delta) = B^1(\delta)$.

The upper limit of δ denoted by θ beyond which the initial optimal basis ceases to be optimal is given by the following criterion.

(A) First consider the case when the vector A_p is not in the basis B^1. Since $B^1(\delta)$ does not change as the value of δ is increased, therefore by substituting δ in place of Φ one gets $B^1(\delta)$ as optimum feasible for $0 \leqslant \delta \leqslant \gamma - \Phi$ if $t_p^1(\Phi' + \delta) \geqslant 0$ for $\Phi' = 0$

i.e. if $Z_{2B}^1(\Phi' + \delta)\,[\bar{Z}_{1p}^1(\Phi') + \delta\,\hat{Z}_{1p}(\Phi') - c_p] - Z_{1B}^1(\Phi' + \delta)$

$$Z_{2p}^1(\Phi') + \delta\hat{Z}_{2p}(\Phi') - d_p].$$

But $Z_{2B}^1(\Phi' + \delta) = Z_{2B}^1(\Phi')$, $Z_{1B}^1(\Phi' + \delta) = Z_{1B}^1(\Phi')$ as the vector A_p is not in the basis

$$\therefore t_p^1(\Phi' + \delta) = Z_{2B}^1(\Phi')\,[\bar{Z}_{1p}^1(\Phi') - C_p] - Z_{1B}^1(\Phi')[\bar{Z}'_{2p}(\Phi') - d_p] +$$

$$+ \delta(Z'_{2B}(\Phi)\hat{Z}_{1p}^1(\Phi') - Z'_{1B}(\Phi')\hat{Z}_{2p}^1(\Phi)) \geqslant 0$$

$$t_p^1(\Phi' + \delta) = t_p^1(\Phi') + \delta\,[Z_{2B}^1(\Phi')\hat{Z}'_{1p}(\Phi') - Z'_{1B}(\Phi')\hat{Z}'_{2p}(\Phi')) \geqslant 0.$$

But $B^1(\Phi') = B^1(\Phi)$ for $\Phi = \Phi' + \delta = \Phi'$(Initially) is the optimal basis $\therefore t_p^1(\Phi) \geqslant 0$.

Thus $B^1(\delta)$ is optimum feasible if $Z_{2B}^1(\Phi')\hat{Z}_{1p}^1(\Phi') - Z_{1B}^1(\Phi')\hat{Z}_{2p}^1(\Phi') \geqslant 0$.

If $Z^1_{2B}(\Phi')\hat{Z}_{1p}(\Phi') - Z^1_{1B}(\Phi')\hat{Z}^1_{2p}(\Phi') < 0$ then the upper limit on δ is given by θ as

$$\theta = \frac{-t^1_p(\Phi')}{Z^1_{2B}(\Phi')\hat{Z}^1_{1p}(\Phi') - Z^1_{1B}(\Phi')\hat{Z}_{2p}(\Phi')}.$$

(B) Next consider the case when A_p is a basis vector.

Let $\bar{B}_p = \bar{A}_p$ and $\hat{B}_p = \hat{A}_p$

$B^1(\Phi') = [B_1 B_2 \ldots \bar{B}_p \ldots B_M]$

$B^1(\delta) = B^1(\Phi')T^1(\delta)$.

Also $B^1(\delta) = B^1(\Phi') + \delta[O \ldots \hat{B}_p O \ldots O]$ where O is a null vector.

$$\therefore\ T^1(\delta) = [B^1(\Phi')]^{-1} B^1(\delta) = [B^1(\Phi')]^{-1}[B^1(\Phi') + \delta[O \ldots \hat{B}_p O \ldots O]]$$

$$= \mathrm{I} + \delta[B^1(\Phi)]^{-1}[O \ldots \hat{B}_p O \ldots O],\ \mathrm{I} \text{ being a unit matrix.}$$

Let β^i be the ith row vector of $[B^1(\Phi')]^{-1}$ and define $y^1_{ip}(\Phi') = \beta^i \hat{B}_p$ then $T^1(\delta)$ is given by

$$\begin{bmatrix} 1 & 0 \ldots 0 \ldots \delta y^1_{1p}(\Phi') \ldots 0 \ldots . 0 \\ 0 & 1 \ldots 0 \ldots \delta y^1_{2p}(\Phi') \ldots 0 \ldots . 0 \\ 0 & 0 \ldots 0\, \mathrm{I} + \delta y^1_{pp}(\Phi') \ldots 0 \ldots . 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \quad 0 \quad \delta y^1_{mp}(\Phi') \quad 0 \quad 1 \end{bmatrix}$$

from which one observes that $T^1(\delta)$ is non-singular if $1 + \delta y^1_{pp}(\Phi') \neq 0$ and $[T^1(\delta)]^{-1}$ is given by

$$[T^1(\delta)]^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dfrac{-\delta y^1_{1p}(\Phi')}{1+\delta y^1_{pp}(\Phi')} & 0 & 0 \\ 0 & 1 & 0 & \dfrac{-\delta y^1_{2p}(\Phi')}{1+\delta y^1_{pp}(\Phi')} & 0 & 0 \\ 0 & 0 & & \dfrac{1}{1+\delta y^1_{pp}(\Phi')} & 0 & 0 \\ 0 & & & \dfrac{-\delta y_{mp}(\Phi')}{1+\delta y^1_{pp}(\Phi')} & 0 & 1 \end{bmatrix}$$

$$\therefore [B^1(\delta)]^{-1} = [T^1(\delta)]^{-1}\,[B^1(\Phi')]^{-1}.$$

Substituting the value of $[B^1(\delta)]^{-1}$ in (1.2) for $k = 1$ one gets

$$X_B^1(\Phi' + \delta) = [T^1(\delta)]^{-1} X_B^1(\Phi')$$

$$Y_j^1(\Phi' + \delta) = [T^1(\delta)]^{-1} Y_j^1(\Phi')$$

$$Z_{1j}^1(\Phi^1 + \delta) = C_B^1[T^1(\delta)]^{-1} y_j^1(\Phi') \tag{1.3}$$

$$Z_{2j}^1(\Phi' + \delta) = d_B^1[T^1(\delta)]^{-1} Y_j^1(\Phi')$$

$$t_j^1(\Phi' + \delta) = \{Z_{2B}^1(\Phi' + \delta)[Z_{1j}^1(\Phi' + \delta) - C_j] - Z_{1B}^1(\Phi' + \delta)$$

$$(Z_{2j}^1(\Phi' + \delta) - d_j)\}.$$

$\therefore$ The new basis is feasible if $X_B^1(\Phi' + \delta) \geqslant 0$ i.e.

$$X'_{Bi} - \left[\frac{\delta y_{ip}^1(\Phi') X_{Bp}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')}\right] \geqslant 0, \quad i \neq p$$

$$\Rightarrow \delta \leqslant \frac{X_{Bi}^1}{X_{Bp}^1 y_{ip}^1(\Phi') - X'_{Bi} y_{pp}^1(\Phi')}$$

$$\text{if } X_{Bp}^1 y_{ip}^1(\Phi') - X_{Bi}^1 Y_{pp}^1(\Phi') > 0$$

and

$$\frac{X_{Bp}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')} \geqslant 0, \quad i = p$$

$$\therefore X_{Bp}^1(\Phi') \geqslant 0 \quad \therefore \; 1 + \delta y_{pp}^1(\Phi') > 0 \Rightarrow \delta < \frac{-1}{y_{pp}^1(\Phi'),}$$

$$\text{if } y_{pp}^1(\Phi') < 0.$$

$$\text{Let } \xi_1' = \begin{cases} \dfrac{-1}{y_{pp}^1(\Phi')} & \text{if } y_{pp}^1(\Phi') < 0 \\ \infty & \text{if } y_{pp}^1(\Phi') \geqslant 0 \end{cases}$$

$$\xi_2' = \min_i \left\{ \frac{X_{Bi}^1}{X_{Bp}^1 y_{ip}^1(\Phi') - X_{Bi}^1 y_{pp}^1(\Phi')} \right.$$

$$\text{if } X_{Bp}^1 y_{ip}^1(\Phi') - X_{Bi}^1 y_{pp}^1(\Phi') > 0.$$

Also the new basis is optimal if $t_j^1(\Phi' + \delta) \geqslant 0 \qquad \forall\, j = 1,2, \ldots n$

i.e.

$$t_j^1(\Phi' + \delta) = \{Z_{2B}^1(\Phi' + \delta)\,(Z_{1j}^1(\Phi' + \delta) - C_j) - Z_{1B}^1(\Phi' + \delta)\,(Z_{2j}^1(\Phi' + \delta) - d_j)\} \geqslant 0.$$

$$\Rightarrow \left\{\left(Z_{2B}^1(\Phi') - \frac{X_{Bp}^1 \delta \hat{Z}_{2p}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')}\right)\left(Z_{1j}^1(\Phi') - C_j - \frac{\delta y_{pj}^1(\Phi')\hat{Z}_{1p}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')}\right)\right.$$

$$\left. - \left(Z_{1B}^1(\Phi') - \frac{X_{Bp}^1 \delta \hat{Z}_{1p}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')}\right)\left(Z_{2j}'(\Phi') - d_j - \frac{\delta y_{pj}'(\Phi')\hat{Z}_{2p}^1(\Phi')}{1 + \delta y_{pp}^1(\Phi')}\right)\right\} \geqslant 0$$

$$\Rightarrow \quad \left\{\left(Z_{2B}^1(\Phi')(Z_{1j}^1(\Phi') - C_j) - Z_{1B}^1(\Phi')\,(Z_{2j}^1(\Phi') - d_j)\right) - \frac{\delta}{1 + \delta y_{pp}^1(\Phi')} \times\right.$$

$$\times (X_{Bp}'\hat{Z}_{2p}^1(\Phi')\,(Z_{1j}'(\Phi') - C_j) - X'_{Bp}\hat{Z}_{1p}'(\Phi')\,(Z_{2j}'(\Phi') - d_j) +$$

$$\left. + Z_{2B}'(\Phi')y_{pj}'(\Phi')\hat{Z}'_{1p}(\Phi') - Z_{1B}'(\Phi')y_{pj}'(\Phi')\hat{Z}_{2p}'(\Phi'))\right\} \geqslant 0$$

$$\Rightarrow \quad t_j'(\Phi') - \delta\mu_j \geqslant 0 \qquad \forall\ j = 1,2 \ldots n,$$

where $\mu_j = [\, - t_j^1(\Phi')y_{pp}^1(\Phi') + X_{Bp}^1(\hat{Z}_{2p}^1(\Phi')\,(Z_{1j}^1(\Phi') - C_j) - \hat{Z}_{1p}^1(\Phi')$

$$(Z_{2j}^1(\Phi') - d_j)) + y_{pj}^1(\Phi')\,(Z_{2B}^1(\Phi')\hat{Z}'_{1p}(\Phi') - \hat{Z}_{1B}^1(\Phi')\hat{Z}_{2p}^1(\Phi')]$$

$$\Rightarrow \quad \delta \leqslant \frac{t_j^1(\Phi')}{\mu_j} \text{ if } \mu_j > 0.$$

Let $\xi_3^1 = \underset{j}{min}\left\{\frac{t_j^1(\Phi')}{\mu_j}, \mu_j > 0\right\}$.

Let $\theta = min\ \{\xi_1^1,\ \xi_2^1,\ \xi_3^1\}$.

Then θ defines the upper limit of δ as follows:

(i) $[T^1(\delta)]^{-1}$ exists if $\theta = \xi_1'$ then at $\delta = \xi_1' = \theta$ the given basis ceases to be basic.

(ii) The fact that $X_B^1(\Phi' + \delta) \geqslant 0 \Rightarrow \theta = \xi_2'$, then for $\delta > \theta$ the given basis no longer remains feasible.

(iii) Again at $\theta = \xi_3' = \delta$ the basis is optimal but as δ increases beyond θ the basis ceases to be optimal.

When (i) holds the condition of parametric programming, problem is termed as in-stable.

Assume that $\theta \neq \xi_1'$. Let $\Phi^2 = \Phi' + \theta$, the new optimal feasible basis for $\Phi = \Phi^2 + \delta$ where $0 \leqslant \delta < \in$ may be obtained by applying either primal or dual simplex method to the current basis.

Writing

$$X_B^2(\Phi^2+\delta)=[B^2(\Phi^2+\delta)]^{-1}b$$

$$Y_j^2(\Phi^2+\delta)=[B^2(\Phi^2+\delta)]^{-1}A_j \tag{1.4}$$

$$Z_{1j}^2(\Phi^2+\delta)=C_B^2[B^2(\Phi^2+\delta)]^{-1}A_j,\ Z_{2j}^2(\Phi^2+\delta)=d_B^2[B^2(\Phi^2+\delta)]^{-1}A_j$$

$$t_j^2(\Phi^2+\delta)=Z_{2B}^2(\Phi^2+\delta)\,(Z_{1j}^2(\Phi^2+\delta)-C_j)-Z_{1B}^2(\Phi^2+\delta)$$

$$[Z_{2j}^2(\Phi^2+\delta)-d_j]$$

as equivalent to (1.3), where C_B^2, d_B^2, Z_{1B}^2, Z_{2B}^2 are the new values of $C_B^1(\Phi')$, $d_B^1(\Phi')$ $Z_{1B}^1(\Phi')$ $Z_{2B}^1(\Phi')$ respectively.

The transformation to the new basis can be brought in the following manner.

$$\text{Let } B^1(\Phi^2+\delta)=B^1(\Phi')\,T^1(\Phi^2+\delta)$$

$$B^2(\Phi^2+\delta)=B^1(\Phi^2+\delta)\,(P(\Phi^2+\delta).$$

Where $P(\Phi^2+\delta)$ is the matrix that transforms a basis to another in the revised simplex method. Thus $P(\Phi^2+\delta)$ is non-singular and $T^1(\Phi^2+\delta)$ is assumed to be non-singular. This assumption can be made true by taking arbitrary small value of δ.

$$\text{Then } [B^1(\Phi^2+\delta)]^{-1}=[T^1(\Phi^2+\delta)]^{-1}[B^1(\Phi')]^{-1}$$

and

$$[B^2(\Phi^2+\delta)]^{-1}=[P(\Phi^2+\delta)]^{-1}[B^1(\Phi^2+\delta)]^{-1}= \tag{1.5}$$

$$=[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}[B^1(\Phi')]^{-1}$$

$$\therefore X_B^2(\Phi^2+\delta)=[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}[B^1(\Phi')]^{-1}b=[P(\Phi^2+\delta)]^{-1}$$

$$[T^1(\Phi^2+\delta)]^{-1}X_B^1(\Phi')$$

$$Y_j^2(\Phi^2+\delta)=[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}Y_j^1(\Phi')$$

$$Z_{1j}^2(\Phi^2+\delta)=C_B^2[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}Y_j^1(\Phi')$$

$$Z_{2j}^2(\Phi^2+\delta)=d_B^2[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}Y_j^1(\Phi')$$

and

$$t_j^2(\Phi^2+\delta)=\{Z_{2B}^2(\Phi^2+\delta)\,(Z_{1j}^2(\Phi^2+\delta)-C_j)-Z_{1B}^2(\Phi^2+\delta)$$

$$(Z_{2j}^2(\Phi^2+\delta)_j-d_j)\}$$

where $X_B^1(\Phi')=[B^1(\Phi')]^{-1}b \qquad Y_j^1(\Phi')=[B^1(\Phi')]^{-1}A_j$.

Another set of transformations equivalent to the above mentioned transformations can be written as;

$$B^1(\Phi^2)=B^1(\Phi')\,T^1(\Phi^2)$$

$$B^2(\Phi^2)=B^1(\Phi')\,P(\Phi^2)$$

$$B^2(\Phi^2+\delta)=B^2(\Phi^2)\,T^2(\delta)$$

where $P(\Phi^2)$ is the transforming matrix for the revised simplex method and $T^1(\Phi^2)$ is equivalent to $T^1(\Phi')$ for the new basis. $T^1(\Phi')$, $P(\Phi^2)$, $T^2(\delta)$ are non-singular.

$$\text{Now } [B^2(\Phi^2)]^{-1} = [P(\Phi^2)]^{-1}\,[B^1(\Phi^2)]^{-1} = [P(\Phi^2)]^{-1}\;T^1(\Phi^2)[B^1(\Phi')]^{-1} \tag{1.6}$$

and

$$[B^2(\Phi^2+\delta)]^{-1} = [T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\,[T^1(\Phi^2)]^{-1}\,[B(\Phi')]^{-1}$$

and

$$\begin{aligned}
X^2_B(\Phi^2+\delta) &= [T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\,[T^1(\Phi^2)]^{-1}\,X^1_B(\Phi')\\
Y^2_j(\Phi^2+\delta) &= [T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\,[T^1(\Phi^2)]^{-1}\,Y^1_j(\Phi') \qquad (1.7)\\
Z^2_{1j}(\Phi^2+\delta) &= C^2_B\,[T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\,[T^1(\Phi^2)]^{-1}\,Y^1_j(\Phi')\\
Z^2_{2j}(\Phi^2+\delta) &= d^2_B[T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\,[T^1(\Phi^2)]^{-1}\,Y^1_j(\Phi')\\
t^2_j(\Phi^2+\delta) &= \{Z^2_{2B}\,(\Phi^2+\delta)\,[Z^2_{1j}(\Phi^2+\delta) -\\
&\quad - C_j) - Z^2_{1B}\,(\Phi^2+\delta)\,[Z^2_{2j}(\Phi^2+\delta) - d_j)\}.
\end{aligned}$$

Comparing the two values of $[B^2(\Phi^2+\delta)]^{-1}$ given by (1.5) and (1.6) one gets:

$$[P(\Phi^2+\delta)]^{-1}[T^1(\Phi^2+\delta)]^{-1}[B^1(\Phi')]^{-1} = [T^2[\delta)]^{-1}[P(\Phi^2)]^{-1}[T^1(\Phi^2)]^{-1}[B^1(\Phi')]^{-1}$$

i.e.

$$[P(\Phi^2+\delta)]^{-1}\,[T^1(\Phi^2+\delta)]^{-1} = [T^2(\delta)]^{-1}\;[P(\Phi^2)]^{-1}\;[T^{\,1}(\Phi^{\,2})]^{-1}.$$

These transformations provide different bases for the same parameter Φ^2. The first set of transformations indicate if $\theta = \xi^1_3$ primal simplex algorithm is used to obtain the new basis for $0 \leqslant \delta$ where as the second set of transformations tell that if $0 = \xi^1_2$ dual simplex algorithm is to be applied to get the new basis. Note that for $0 < \delta < \in$ $B^1(\Phi^2+\delta)$ is infeasible if $\theta = \xi^1_2$ and not optimal if $\theta = \xi^1_3$.
Suppose further one likes to find the upperbound for δ for which $B^2(\Phi^2+\delta)$ ceases to be optimum feasible. The first set of transformations do not help in determining the upper bound on δ. For that one has to use the second set of transformations.

Let

$$\begin{aligned}
X^1_B(\Phi^2) &= [T^1(\Phi^2)]^{-1}\,X^1_B(\Phi')\\
Y^1_j(\Phi^2) &= [T^1(\Phi^2)]^{-1}\,Y^1_j(\Phi')\\
Z^1_{1j}(\Phi^2) &= C^1_B[T^1(\Phi^2)]^{-1}\,Y^1_j(\Phi') \qquad (1.8)\\
Z^1_{2j}(\Phi^2) &= d^1_B[T^1(\Phi^2)]^{-1}Y^1_j(\Phi')
\end{aligned}$$

and

$$
\begin{aligned}
X_B^2(\Phi^2) &= [P(\Phi^2)]^{-1}\, X_B^1(\Phi^2) \\
Y_j^2(\Phi^2) &= [P(\Phi^2)]^{-1}\, Y_j^1(\Phi^2) \\
Z_{1j}^2(\Phi^2) &= C_B^2[P(\Phi^2)]^{-1} Y_j^1(\Phi^2) \qquad (1.9) \\
Z_{2j}^2(\Phi^2) &= d_B^2[P(\Phi^2)]^{-1}\ Y_j^1(\Phi^2).
\end{aligned}
$$

The set of transformations (1.8) determine optimum feasible first basis for $\Phi = \Phi^2$ and equation (1.9) gives optimum feasible second basis for $\Phi = \Phi^2$ and $\delta = 0$. Substituting the values obtained from (1.9) in (1.7)

$$
\begin{aligned}
X_B^2(\Phi^2 + \delta) &= [T^2(\delta)]^{-1}\, X_B^2(\Phi^2) \\
Y_j^2(\Phi^2 + \delta) &= [T^2(\delta)]^{-1}\, Y_j^2(\Phi^2)
\end{aligned}
$$

$$
Z_{1j}^2(\Phi^2 + \delta) = C_B^2[T^2(\delta)]^{-1}\, Y_j^2(\Phi^2) = C_B^2[T^2(\delta)]^{-1}\,[P(\Phi^2)]^{-1}\ [T(\Phi^2)]^{-1}\, Y_j^1(\Phi')
$$

$$
Z_{2j}^2(\Phi^2 + \delta) = d_B^2[T^2(\delta)]^{-1}\, Y_j^2(\Phi^2) \qquad (1.10)
$$

$$
t_j^2(\Phi^2 + \delta) = \{Z_{2B}^2(\Phi^2 + \delta)[Z_{1j}^2(\Phi^2 + \delta) - C_j] - Z_{1B}^2(\Phi^2 + \delta)[Z_{2j}^2(\Phi^2 + \delta) - d_j]\}.
$$

Comparing (1.10) with (1.3) one observes that these results are identical except that there is a difference in the superscript, therefore the upper limt θ^2 on δ will be determined exactly in the same manner as has been worked out for the superscript 1.

Let $\Phi^3 = \Phi^2 + \theta^2$, since $B^2(\delta)$ is optimum feasible for $0 \leqslant \delta \leqslant \theta^2\, B^2\ (\Phi)$ is optimum feasible for $\Phi^2 \leq \theta \leq \Phi^3$. Using principle of Mathematical Induction to increase Φ further beyond Φ^3, Φ^4 can be obtained as Φ^2 was obtained from Φ^1 by the parametrization process.

SECTION II

In this section the parametric algorithm is developed along with a numerical example to explain the above theory.

Let the initial solution for $\Phi = 0$ be given by $X_B^k\ (\Phi)$ and let $Y_j^k(\Phi)$, $Z_{1j}^k(\Phi\ )$, $Z_{2j}^k(\Phi)$ be the corresponding values of Y_j, Z_{1j}, Z_{2j} at $\Phi = 0$.

Since these terms correspond to the elements of the same simplex tableau, its elements are denoted by $E_{1j}^k\ (\Phi)$ where the suffices i and j indicate a row and a column, respectively. In this respect, three columns are used for $j = p$ in order to distinguish $y_p^k(\Phi)$, $\bar{y}_p^k(\Phi)$, $\hat{y}_p^k(\Phi)$ and $Z_{ip}^k(\Phi)$, $\bar{Z}_{ip}^k(\Phi)$ $\hat{Z}_{ip}\ (\Phi)$ for $i = 1,2$. Initially set $k = 1$ and $t = 2$.

The algorithm is given in the following steps:

(1) If A_p is in the basis go to (2) otherwise go to (6).

(2) If $\hat{t}_p(\Phi^k) \geqslant 0$ Stop (implies that the solution is optimum for all

$$
\Phi^k \leqslant \qquad \Phi \leqslant \gamma \text{ otherwise go to step (3).}
$$

(3) Determine θ^k and set $\Phi^t = \Phi^k + \theta^k$, and obtain $E_{ij}^k(\Phi^t)$ for all i and j, go to step (4).

(4) If $y_{ip}^k\ (\Phi^t) \leqslant 0$ for all i stop (implies that the problem has an unbounded solution for all $\Phi^t \leqslant \Phi \leqslant \gamma$ or the feasible region of Φ is disconnected) otherwise go to (5).

(5) Obtain $E_{ij}^k(\Phi^t)$ for all i and j by the primal simplex method.

Let $k = k + 1$ and $t = t + 1$ and go to step (6).

(6) Determine θ^k and set $\Phi^t = \Phi^k + \theta^k$ and go to (7).

(7) If $\theta^k = \xi_1^k$, stop (implies that the problem became instable at $\Phi = \Phi^t$) otherwise go to (8).

(8) Obtain $E_{ij}^k(\Phi^t)$ for all i and j. Go to (9).

(9) If $\theta^k = \xi_2^k$ go to (10). Otherwise go to (12).

(10) If $y_{gj}^k(\Phi^t) \leqslant 0$ for all j where g corresponds to the basis vector to be removed, stop (implies that the problem either does not have a feasible solution for all $\Phi^t \leqslant \Phi \leqslant \gamma$ or the feasible region of Φ is disconnected). Otherwise go to (11).

(11) Obtain $E_{ij}^k(\Phi^t)$ for all i and j by the dual simplex algorithm. Go to (14).

(12) If $y_{ih}^k(\Phi^t) \geqslant 0$ for all i where h corresponds to the vector to be brought into the basis stop (implies that the problem has an unbounded solution for all $\Phi^t \leqslant \Phi \leqslant \gamma$ or the feasible region of Φ is disconnected). Otherwise go to (13).

(13) Obtain $E_{ij}^k(\Phi^t)$ for all i and j by the primal simplex algorithm and go to (14).

(14) If $\Phi^t \geqslant \gamma$ stop (implies that the parametrization process ends) otherwise go to (15).

(15) Let $k = k + 1$ and $t = t + 1$ and go to (1).

It is assumed that the solution is non-degenerate and θ^k is uniquely determined.

Now consider the case when θ^k is not unique. First suppose that $\theta^k = \xi_2^k$ but ξ_2^k is not uniquely determined. Then if the new basis is obtained for $\Phi = \Phi^t$, it will be primal degenerate. The new basis may be feasible only for $\Phi = \Phi^t$ and infeasible for $\Phi > \Phi^t$.
But by changing the basis at $\Phi = \Phi^t$ it is possible to obtain a basis which is not primal degenerate and in turn, the value of Φ can be increased further.

Secondly, suppose that $\theta^k = \xi_3^k$ and ξ_3^k is not uniquely determined. Then if the new basis is obtained for $\Phi = \Phi^t$ the new basis will be dual degenerate. However again by changing the basis at $\Phi = \Phi^t$ it is possible to obtain a basis that is not dual degenerate and the value of Φ can be increased further. Third suppose that $\theta^k = \xi_2^k = \xi_1^k$ or $\theta^k = \xi_3^k = \xi_1^k$, or $\theta^k = \xi_2^k = \xi_3^k = \xi_1^k$. If any of these situations occurs the parametrization process should be stopped since the basis becomes instable at $\Phi = \Phi^t$.

Numerical Example

Consider the parametric linear fractional functional programming problem viz. Maximize

$$\frac{Z_1}{Z_2} = \frac{X_1 + 2X_2 + X_3 + 1}{2X_1 + X_2 + 3X_3 + 2}$$

Subject to

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} X_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Phi X_2 + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} X_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} X_4 +$$

$$+ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} X_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} X_6 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$X_i \geqslant 0$ $i = 1,2,3,4,5,6$ where X_4, X_5, X_6 are slack variables.
The simplex tables are given as under

Table I

		C_j	1	2	2	0	1	0	0	0	
		d_j	2	1	1	0	3	0	0	0	
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\hat{X}_2$	$\dot{X}_3$	X_4	X_5	X_6	b
0	0	X_4	1	1	1	0	2	1	0	0	3
0	0	X_5	1	$2+\Phi$	2	1	1	0	1	0	5
0	0	X_6	0	$\boxed{2-\Phi}$	2	−1	2	0	0	1	4
$Z_{1B}=1$	$Z_{1j}-C_j$		−1	−2	−2	0	−1	0	0	0	
$Z_{2B}=2$	$Z_{2j}-d_j$		−2	−1	−1	0	−3	0	0	0	
	t_j		0	−3	−3	0	1	0	0	0	

Table II

		C_j	1	2	2	0	1	0	0	0		
d_B	C_B	d_j	2	1	1	0	3	0	0	0		
		X_B	X_1	X_2	$\overline{X}_2$	$\widehat{X}_2$	X_3	X_4	X_5	X_6	b	For $\Phi=0$ b_0
0	0	X_4	$\frac{-1-\Phi}{2-\Phi}$	0	$\frac{-\Phi}{2-\Phi}$	$\frac{1}{2-\Phi}$	$\frac{2(1-\Phi)}{2-\Phi}$	1	0	$\frac{-1}{2-\Phi}$	$\frac{2-3\Phi}{2-\Phi}$	1
0	0	X_5	$\frac{-4-4\Phi}{2-\Phi}$	0	$\frac{-4\Phi}{2-\Phi}$	$\frac{4}{2-\Phi}$	$\frac{-2-3\Phi}{2-\Phi}$	0	1	$\frac{-(2+\Phi)}{2-\Phi}$	$\frac{2-9\Phi}{2-\Phi}$	1
1	2	X_2	$\frac{3}{2-\Phi}$	1	$\frac{2}{2-\Phi}$	$\frac{-1}{2-\Phi}$	$\frac{2}{2-\Phi}$	0	0	$\frac{1}{2-\Phi}$	$\frac{4}{2-\Phi}$	2
$Z_{1B}=\frac{10-\Phi}{2-\Phi}$	$Z_{1j}-C_j$		$\frac{4+\Phi}{2-\Phi}$	0	$\frac{2\Phi}{2-\Phi}$	$\frac{-2}{2-\Phi}$	$\frac{2+\Phi}{2-\Phi}$	0	0	$\frac{2}{2-\Phi}$		
$Z_{2B}=\frac{8-2\Phi}{2-\Phi}$	$Z_{2j}-d_j$		$\frac{-1+2\Phi}{2-\Phi}$	0	$\frac{\Phi}{2-\Phi}$	$\frac{-1}{2-\Phi}$	$\frac{-4+3\Phi}{2-\Phi}$	0	0	$\frac{1}{2-\Phi}$		
		l_j	$\frac{21}{2-\Phi}$	0	$\frac{3\Phi}{2-\Phi}$	$\frac{-3}{2-\Phi}$	$\frac{28-\Phi}{2-\Phi}$	0	0	$\frac{3}{2-\Phi}$		

The initial optimum feasible solution for $\Phi = 0$ is given in Table II. From table II one gets $\xi_1' = 2$, $\xi_2' = \frac{2}{9}$ and $\xi_3' = 28$ $\therefore$ $min\{(\xi_1', \xi_2', \xi_3')\} = \frac{2}{9}$ $\therefore$ The initial solution is optimal for $0 \leqslant \Phi \leqslant 2/9$. Replacing Φ by $\Phi' + \frac{2}{9}$ Table II can be rewritten as

Table (II)'

		C_j	1	2	2	0	1	0	0	0		For
		d_j	2	1	1	0	3	0	0	0		$\Phi = \frac{2}{9}$
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\hat{X}_2$	X_3	X_4	X_5	X_6	b	b_0
0	0	X_4	$\frac{-\frac{11}{9} - \Phi^1}{\frac{16}{9} - \Phi^1}$	0	$\frac{-\left(\frac{2}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	$\frac{1}{\frac{16}{9} - \Phi^1}$	$\frac{2\left(\frac{7}{9} - \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	1	0	$\frac{-1}{\frac{16}{9} - \Phi^1}$	$\frac{3\left(\frac{4}{9} - \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	$\frac{3}{4}$
0	0	X_5	$\frac{-4\left(\frac{11}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	0	$\frac{-4\left(\frac{2}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	$\frac{4}{\frac{16}{9} - \Phi^1}$	$\frac{3\left(\frac{8}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	0	1	$\boxed{\frac{-\left(\frac{20}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}}$	$\frac{-9\Phi^1}{\frac{16}{9} - \Phi^1}$	0
1	2	X_2	$\frac{3}{\frac{16}{9} - \Phi^1}$	1	$\frac{2}{\frac{16}{9} - \Phi^1}$	$\frac{-1}{\frac{16}{9} - \Phi^1}$	$\frac{2}{\frac{16}{9} - \Phi^1}$	0	0	$\frac{1}{\frac{16}{9} - \Phi^1}$	$\frac{4}{\frac{16}{9} - \Phi^1}$	$\frac{9}{4}$
$Z_{1B} = \frac{\frac{88}{9} - \Phi_1}{\frac{16}{9} - \Phi_1}$		$Z_{1j} - C_j$	$\frac{\frac{38}{9} + \Phi^1}{\frac{16}{9} - \Phi^1}$	0	$\frac{\frac{4}{9} + 2\Phi^1}{\frac{16}{9} - \Phi^1}$	$\frac{-2}{\frac{16}{9} - \Phi^1}$	$\frac{\frac{20}{9} + \Phi^1}{\frac{16}{9} - \Phi^1}$	0	0	$\frac{2}{\frac{16}{9} - \Phi^1}$		
$Z_{2B} = \frac{\frac{68}{9} - 2\Phi^1}{\frac{16}{9} - \Phi^1}$		$Z_{2j} - d_j$	$\frac{-\frac{5}{9} + 2\Phi^1}{\frac{16}{9} - \Phi^1}$	0	$\frac{\frac{2}{9} + \Phi^1}{\frac{16}{9} - \Phi^1}$	$\frac{-1}{\frac{16}{9} - \Phi^1}$	$\frac{3\left(\frac{-10}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	0	0	$\frac{1}{\frac{16}{9} - \Phi^1}$		
		t_j	$\frac{21}{\frac{16}{9} - \Phi^1}$	0	$\frac{3\left(\frac{2}{9} + \Phi^1\right)}{\frac{16}{9} - \Phi^1}$	$\frac{-3}{\frac{16}{9} - \Phi^1}$	$\frac{\frac{250}{9} - \Phi^1}{\frac{16}{9} - \Phi^1}$	0	0	$\frac{3}{\frac{16}{9} - \Phi^1}$		

Applying dual simplex method $\therefore \theta = 0 = min\ \{\xi_1^2,\ \xi_2^2,\ \xi_3^2\} = mtn \left\{\frac{16}{9},\ 0,\ \frac{9}{250}\right\}$ The new value of Φ is $\Phi^2 = \Phi^2 + 0 = \Phi^2$

Table III

		C_j	1	2	2	0	1	0	0	0		For
		d_j	2	1	1	0	3	0	0	0		$\Phi^2 = 0$
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\widehat{X}_2$	X_3	X_4	X_5	X_6	b	b_0
0	0	X_4	$\frac{\frac{11}{9}+\Phi^2}{\frac{20}{9}+\Phi^2}$	0	$\frac{\frac{2}{9}+\Phi^2}{\frac{20}{9}+\Phi^2}$	$\frac{-1}{\frac{20}{9}+\Phi^2}$	$\frac{\frac{2\Phi^2}{1}+\frac{31}{9}}{\frac{20}{9}+\Phi^2}$	1	$\frac{-1}{\frac{20}{9}+\Phi^2}$	0	$\frac{3\left(\frac{5}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	$\frac{3}{4}$
0	0	X_6	$\frac{4\left(\frac{11}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	0	$\frac{4\left(\frac{2}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	$\frac{-4}{\frac{20}{9}+\Phi^2}$	$\frac{3\left(\frac{8}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	0	$\frac{-\left(\frac{16}{9}-\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	1	$\frac{9\Phi^2}{\frac{20}{9}+\Phi^2}$	0
1	2	X_2	$\frac{1}{\frac{20}{9}+\Phi^2}$	1	$\frac{2}{\frac{20}{9}+\Phi^2}$	$\frac{1}{\frac{20}{9}+\Phi^2}$	$\frac{1}{\frac{20}{9}+\Phi^2}$	0	$\boxed{\frac{1}{\frac{20}{9}+\Phi^2}}$	0	$\frac{5}{\frac{20}{9}+\Phi^2}$	$\frac{9}{4}$
$Z_{1B} = \frac{\frac{110}{9}+\Phi^2}{\frac{20}{9}+\Phi^2}$		$Z_{1j} - C_j$	$\frac{\frac{-2}{9}-\Phi^2}{\frac{20}{9}+\Phi^2}$	0	$\frac{2\left(\frac{-2}{9}-\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	$\frac{2}{\frac{20}{9}+\Phi^2}$	$\frac{\frac{-2}{9}-\Phi^2}{\frac{20}{9}+\Phi^2}$	0	$\frac{2}{\frac{20}{9}+\Phi^2}$	0		
$Z_{2B} = \frac{\frac{85}{9}+2\Phi^2}{\frac{20}{9}+\Phi^2}$		$Z_{2j} - d_j$	$\frac{\frac{-31}{9}-2\Phi^2}{\frac{20}{9}+\Phi^2}$	0	$\frac{\frac{-2}{9}-\Phi^2}{\frac{20}{9}+\Phi^2}$	$\frac{1}{\frac{20}{9}+\Phi^2}$	$\frac{-3\left(\frac{17}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	0	$\frac{1}{\frac{20}{9}+\Phi^2}$	0		
		t_j	$\frac{18}{\frac{20}{9}+\Phi^2}$	0	$\frac{-3\left(\frac{2}{9}+\Phi^2\right)}{\frac{20}{9}+\Phi^2}$	$\frac{3}{\frac{20}{9}+\Phi^2}$	$\frac{\frac{272}{9}+\Phi^2}{\frac{20}{9}+\Phi^2}$	0	$\frac{3}{\frac{20}{9}+\Phi^2}$	0		

To avoid degeneracy introducing X_5 into the basis and removing X_2 from the basis

Table IV

		C_j	1	2	2	0	1	0	0	0	
		d_j	2	1	1	0	3	0	0	0	
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\hat{X}_2$	X_3	X_4	X_5	X_6	b
0	0	X_4	1	1	1	0	2	1	0	0	3
0	0	X_5	3	$\frac{16}{9} - \Phi^2$	2	−1	2	0	1	0	4
0	0	X_6	1	1	2	1	1	0	0	1	5
$Z_{1B} = 1$		$Z_{1j} - C_j$	−1	−2	−2	0	−1	0	0	0	
$Z_{2B} = 1$		$Z_{2j} - d_j$	−2	−1	−1	0	−3	0	0	0	
		t_j	0	−3	−3	0	1	0	0	0	

Table V

		C_j	1	2	2	0	1	0	0	0		
		d_j	2	1	1	0	3	0	0	0		
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\widehat{X}_2$	X_3	X_4	X_5	X_6	b	b_0
0	0	X_4	$\frac{\frac{-11}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	0	$\frac{\frac{-2}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	$\frac{1}{\frac{16}{9} - \Phi^2}$	$\frac{2\left(\frac{7}{9} - \Phi^2\right)}{\frac{16}{9} - \Phi^2}$	1	0	$\frac{-1}{\frac{16}{9} - \Phi^2}$	$\frac{3\left(\frac{4}{9} - \Phi^2\right)}{\frac{16}{9} - \Phi^2}$	$\frac{3}{4}$
1	2	X_2	$\frac{3}{\frac{16}{9} - \Phi^2}$	1	$\frac{2}{\frac{16}{9} - \Phi^2}$	$\frac{-1}{\frac{16}{9} - \Phi^2}$	$\frac{2}{\frac{16}{9} - \Phi^2}$	0	0	$\frac{1}{\frac{16}{9} - \Phi^2}$	$\frac{4}{\frac{16}{9} - \Phi^2}$	$\frac{9}{4}$
0	0	X_5	$\frac{\frac{-11}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	0	$\frac{\frac{14}{9} - 2\Phi^2}{\frac{16}{9} - \Phi^2}$	$\frac{\frac{25}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	$\frac{\frac{-2}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	0	1	$\frac{-1}{\frac{16}{9} - \Phi^2}$	$\frac{\frac{44}{9} - 5\Phi^2}{\frac{16}{9} - \Phi^2}$	$\frac{11}{4}$
$Z_{1B} = \frac{\frac{88}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$		$Z_{1j} - C_j$	$\frac{\frac{38}{9} + \Phi^2}{\frac{16}{9} - \Phi^2}$	0	$\frac{2\left(\frac{2}{9} + \Phi^2\right)}{\frac{16}{9} - \Phi^2}$	$\frac{-2}{\frac{16}{9} - \Phi^2}$	$\frac{\frac{20}{9} + \Phi^2}{\frac{16}{9} - \Phi^2}$	0	0	$\frac{2}{\frac{16}{9} - \Phi^2}$		
$Z_{2B} = \frac{\frac{68}{9} - 2\Phi^2}{\frac{16}{9} - \Phi^2}$		$Z_{2j} - d_j$	$\frac{\frac{-5}{9} + 2\Phi^2}{\frac{16}{9} - \Phi^2}$	0	$\frac{\frac{2}{9} + \Phi^2}{\frac{16}{9} - \Phi^2}$	$\frac{-1}{\frac{16}{9} - \Phi^2}$	$\frac{3\left(\frac{-10}{9} + \Phi^2\right)}{\frac{16}{9} - \Phi^2}$	0	0	$\frac{1}{\frac{16}{9} - \Phi^2}$		
		t_j	$\frac{21}{\frac{16}{9} - \Phi^2}$	0	$\frac{3\left(\frac{2}{9} + \Phi^2\right)}{\frac{16}{9} - \Phi^2}$	$\frac{-3}{\frac{16}{9} - \Phi^2}$	$\frac{\frac{250}{9} - \Phi^2}{\frac{16}{9} - \Phi^2}$	0	0	$\frac{3}{\frac{16}{9} - \Phi^2}$		

$\theta^3 = min\left\{\frac{16}{9}, \frac{4}{9}, \frac{250}{9}\right\} = \frac{4}{9}$ $\therefore$ Replacing Φ^2 by $\Phi^3 + \frac{4}{9}$ Table V can be rewritten as

Table V′

		C_j	1	2	2	0	1	0	0	0	
		d_j	2	1	1	0	3	0	0	0	
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\hat{X}_2$	X_3	X_4	X_5	X_6	b
0	0	X_4	$\frac{\frac{-5}{3} - \Phi^3}{\frac{4}{3} - \Phi^2}$	0	$\frac{\frac{-2}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	$\frac{1}{\frac{4}{3} - \Phi^3}$	$\frac{2\left(\frac{1}{3} - \Phi^3\right)}{\frac{4}{3} - \Phi^3}$	1	0	$\frac{-1}{\frac{4}{3} - \Phi^3}$	$\frac{-3\Phi^3}{\frac{4}{3} - \Phi^3}$ 0
1	2	X_2	$\frac{3}{\frac{4}{3} - \Phi^3}$	1	$\frac{2}{\frac{4}{3} - \Phi^3}$	$\frac{-1}{\frac{4}{3} - \Phi^3}$	$\frac{2}{\frac{4}{3} - \Phi^3}$	0	0	$\frac{1}{\frac{4}{3} - \Phi^3}$	$\frac{4}{\frac{4}{3} - \Phi^3}$ 3
0	0	X_5	$\frac{\frac{-5}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	0	$\frac{\frac{2}{3} - 2\Phi^3}{\frac{4}{3} - \Phi^3}$	$\frac{\frac{7}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	$\frac{\frac{-2}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	0	1	$\frac{-1}{\frac{4}{3} - \Phi^3}$	$\frac{\frac{8}{3} - 5\Phi^3}{\frac{4}{3} - \Phi^3}$ 2
$Z_{1B} = \frac{\frac{84}{9} - \Phi^3}{\frac{4}{3} - \Phi^3}$		$Z_{1j} - C_j$	$\frac{\frac{42}{9} + \Phi^3}{\frac{4}{3} - \Phi^3}$	0	$\frac{2\left(\frac{2}{3} + \Phi^3\right)}{\frac{4}{3} - \Phi^3}$	$\frac{-2}{\frac{4}{3} - \Phi^3}$	$\frac{\frac{8}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	0	0	$\frac{2}{\frac{4}{3} - \Phi^3}$	
$Z_{2B} - \frac{\frac{20}{3} - 2\Phi^3}{\frac{4}{3} - \Phi^3}$		$Z_{2j} - d_j$	$\frac{\frac{1}{3} + 2\Phi^3}{\frac{4}{3} - \Phi^3}$	0	$\frac{\frac{2}{3} + \Phi^3}{\frac{4}{3} - \Phi^3}$	$\frac{-1}{\frac{4}{3} - \Phi^3}$	$\frac{3\left(\frac{-2}{3} + \Phi^2\right)}{\frac{4}{3} - \Phi^3}$	0	0	$\frac{1}{\frac{4}{3} - \Phi^3}$	
		t_j	$\frac{21}{\frac{4}{3} - \Phi^3}$	0	$\frac{3\left(\frac{2}{3} + \Phi^3\right)}{\frac{4}{3} - \Phi^3}$	$\frac{-3}{\frac{4}{3} - \Phi^3}$	$\frac{\frac{82}{3} - \Phi^3}{\frac{4}{3} - \Phi^3}$	0	0	$\frac{3}{\frac{4}{3} - \Phi^3}$	

Again $\theta^4 = min\left\{\frac{4}{3},\ 0,\ \frac{82}{3}\right\} = min\{\xi_{31}^3,\ \xi_{32}^3\ \xi_{33}^3\}$ Replacing Φ^3 by $\Phi^4 + 0$ and applying dual simplex method

Table VI

		C_j	1	2	2	0	1	0	0	0		
		d_j	2	1	1	0	3	0	0	0		
d_B	C_B	X_B	X_1	X_2	$\overline{X}_2$	$\widehat{X}_2$	X_3	X_4	X_5	X_6	b	b_0
0	0	X_6	$\frac{5}{3} + \Phi^4$	0	$\frac{2}{3} + \Phi^4$	-1	$-2\left(\frac{1}{3} - \Phi^4\right)$	$-\left(\frac{4}{3} - \Phi^4\right)$	0	1	$3\Phi^4$	0
1	2	X_2	1	1	1	0	2	1	0	0	3	3
0	0	X_5	0	0	1	1	-1	-1	1	0	2	2
$Z_{1B} = 7$		$Z_{1j} - C_j$	1	0	0	0	3	2	0	0		
$Z_{2B} = 5$		$Z_{2j} - d_j$	-1	0	0	0	-1	1	0	0		
		t_j	12	0	0	0	22	3	0	0		

$\therefore$ The solution is optimal for $0 \leqslant \frac{4}{3} \leqslant \Phi < \infty$ and non-degenerate for every $\Phi = \Phi^4 > 0$ and the optimal value is 7/5 for $\Phi = \frac{4}{3}$.

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LA PROGRAMMATION FONCTIONNELLE-FRACTIONNAIRE AYANT UN PARAMETRE DANS UN VECTEUR D'ACTIVITE

On présente un algorithme qui paramétrise un vecteur d'activité pour un problème de programmation fonctionnelle-fractionnaire linéaire.

Tout d'abord, le problème a été résolu par une méthode qui garde le paramètre Φ au niveau 0. Ensuite, on a cherché plusieurs valeurs du paramètre Φ, à savoir Φ^1 Φ^2, ..., Φ^k et une série de bases qui lui correspondent, B^1, B^2,..., B^k, obtenues de telle manière, pour que B^k soit optimum et possible pour tous les $\Phi^T \leqslant \Phi \leqslant \gamma$, ou bien illimité et impossible pour $0 \leqslant \Phi - \Phi^T \leqslant \varepsilon$ où ε est un nombre réduit, arbitraire et positif et γ est tout nombre réel.

ФУНКЦИОНАЛЬНО-ДРОБНОЕ ПРОГРАММИРОВАНИЕ С ОДНИМ ПАРАМЕТРОМ В ОДНОМ ВЕКТОРЕ-СТОЛБЦЕ

Настоящая работа представляет алгоритм, параметризирующий один вектор — столбец для одной задачи функционально — дробного линейного программирования.

В первую очередь, задача была решена методом сохранения параметра на нулевом уровне. Затем, было начато нахождение более многих значений параметра Φ а именно Φ^1, Φ^2 ..., Φ^k и ряда соответствующих базисов $B^1, B^2, ..., B^k$, реализованных таким образом, что B^k является либо оптимумом и возможным для всех $\Phi^T \leqslant \Phi \leqslant \gamma$, либо безконечным или невозможным для $0 \leqslant \Phi - \Phi^T \leqslant \varepsilon$, где ε является малым числом, произвольным и позитивным, а γ — любым реальным числом.

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