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On the Asymptotic Distribution of a Random Process with a Random Time

The paper dwells on the following problem: Given a sequence of independent random variables $\{X_j, 1 \leqslant j < \infty\}$ *and* $Y_n = \sum_{1=n}^{n} X_j$; *and given a sequence of random variables with positive integer values* $\{\nu_n, 1 \leqslant n < \infty\}$ *find the distribution of the sequence* $\{Y_{\nu_n}, 1 < n < \infty\}$, *that is the distribution of the sum of a random number of random variables.*

In order to solve this problem the authoress forgoes the independence hypothesis, setting smoother conditions either on the common distribution of the variables (Y_n, ν_n), *or on the sort of ergodicity conditions on the sequence* $\{\nu_n, 1 \leqslant n < \infty\}$.

1. By a random process with a random discrete time we understand, in agreement with the terminology used in [2], a process of the form $\{Y_{\nu_n}, 1 \leqslant n + + \infty\}$, where $\{Y_n, 1 \leqslant n_o > \infty\}$ is a sequence of random variables defined on a probability field $\{\Omega, \mathfrak{K}, P\}$, and $\{\nu_n, 1 \leqslant n < + \infty\}$ is a sequence of random variables with positive integer values also defined on $\{\Omega, \mathfrak{K}, P\}$. Researches concerning the asymptotic distribution of such a process have followed two lines. Works following the first line have led to the form of the asymptotic distribution of the random process with random time $\{Y_{\nu_n}, 1 \leqslant n < + \infty\}$, under certain conditions imposed to the sequences of random variables $\{Y_n, 1 \leqslant n < + \infty\}$ and $\{\nu_n, 1 \leqslant n < + \infty\}$. The most general result reached on this line is given in [1]. On following the second line — narrower than the first — we have found conditions under which the asymptotic distribution of the random process $\{Y_n, 1 \leqslant n < + \infty\}$ remains invariant when one passes from the ordinary time, monotonously flowing towards the in-

linity, to the random time. Out of the works complying with the seconp fine we remark [2].

2. Let $\{X_j,\ 1 \leqslant j + \infty\}$ be a sequence of independent random variables defined on a probability field $\{\Omega, \mathfrak{K}, P\}$ and $Y_n = \sum_{j=1}^{n} X_j,\ 1 \leqslant n < + \infty$. The theorem in [1] gives the form of the asymptotic distribution of the random process with a random time $\{Y_{\nu_n}\ 1 \leqslant n < + \infty\}$, suitably normalized, in case that for any n the random variable ν_n is independent of the random variables $X_j, 1 \leqslant j < + \infty$. Further, we prove a theorem similar to the theorem in [1], forgoing the independence hypothesis of the random time ν_n of the independent random variables $X_j,\ 1 \leqslant j < + \infty$.

Theorem 1. Let $\{X_j,\ 1 \leqslant 1 < + \infty\}$ be a sequence of independent random variables, defined on a probability field

$$\{\Omega, \mathfrak{K}, P\},\ Y_n = \sum_{i=1}^{n} X_j,\ 1 \leqslant n < + \infty \text{ and } \{\nu_m,\ 1 \leqslant n < + \infty\}$$

a sequence of random variables with positive integer values, also defined on $\{\Omega, \mathfrak{K}, P\}$. If

$$\lim_{n \to + \infty} P\left(\frac{Y_n - an^{\alpha}}{bn^{\beta}} < x,\ \frac{\nu_n - cn^{\gamma}}{dn^{\delta}} < y\right) = F(x, y) \tag{2.1}$$

in any point of continuity of the non-degenerate distribution function $F(x, y)$, where $\alpha > \beta,\ b \neq 0,\ \gamma > \delta > 0,\ d \neq 0$, then

$$\lim_{n \to + \infty} P\left(\frac{Y\nu_n - gn^{\lambda}}{hn^{\mu}} < x\right) = G(x). \tag{2.2}$$

Let U and V be the random variables defined on the probability field $\{\Omega, \mathfrak{K}, P\}$ having the common distribution function $F(x, y)$. The constants g, h, λ, μ and the distribution function $G(x)$ in relationship (2.2) are to be found as follows:

I. If $a \neq 0,\ c \neq 0$ and $\gamma\beta < \gamma(\alpha - 1) + \delta$ or $a = 0$, but $c \neq 0$, then

$$g = ac^{\alpha},\ h = bc^{\beta},\ \lambda = \gamma\alpha;\ \mu = \gamma\beta,\ G(x) = F(x, + \infty).$$

II. If $a \neq 0,\ c \neq 0$ and $\gamma(\alpha - 1) + \delta > \gamma\beta$, then

$$g = ac^{\alpha},\ h = \alpha\, ac^{\alpha-1}\, d,\ \lambda = \gamma\alpha,\ \mu = \gamma(\alpha - 1) + \delta,\ G(x) = F(+ \infty, x).$$

III. If $a \neq 0,\ c \neq 0$ and $\gamma\ (\alpha - 1) + \delta = \gamma\beta$, then

$$g = ac^{\alpha},\ h = 1,\ \lambda = \gamma\alpha,\ \mu = \gamma\beta,\ G(x) = P\{\alpha ac^{\alpha-1}\, dV + bc^{\beta} U < x\}$$

IV. If $a \neq 0,\ c = 0$, then

$$g = 0,\ h = ad^{\alpha},\ \mu = \alpha\delta,\ G(x) = P\{V^{\alpha} < x\}.$$

V. If $a = 0,\ c = 0$, then

$$g = 0,\ h = bd^{\beta},\ \mu = \beta\,\delta,\ G(x) = P\{UV^{\beta} < x\}.$$

Proof. Taking into account the results in [6] if the sequence of the distribution functions corresponding to the random vectors (ξ_n, η_n), $1 \leqslant n < \infty$, slightly converges towards the distribution function of the random vector (ξ_o, η_o), then a sequence of random vectors $\{(\overline{\xi}_n, \overline{\eta}_n), 0 \leqslant n < +\infty\}$, can be buils up, so that the distribution functions corresponding to the random vectort (ξ_n, η_n) and $(\overline{\xi}_n, \overline{\eta}_n)$ shall coincide for any n, $0 \leqslant n < +\infty$, and the sequence of random vectors $\{(\overline{\xi}_n, \overline{\eta}_n), 1 \leqslant n < +\infty\}$ shall converge pretty surely towards the random vector (ξ_0, η_o). Indeed, for this purpose it will do to choose the probability field $\{\Omega, \mathfrak{K}, P\}$ as follows: $\Omega = [0, 1]$, $\mathfrak{K} = \mathfrak{B}_{[0,1]}$ (τ — the algebra of the Borel sets on $[0, 1]$ and P the Lebesgue measure on $[0, 1]$. Further the reasoning is like in proving the theorem in [1].

Let $\{(U_n, V_n), 1 \leqslant n < +\infty\}$ be a sequence of random vectors having respectively the same distribution functions as the random vectors of the sequence $\left\{\left(\frac{Y_n - an^\alpha}{bn^\beta}, \frac{\nu_n - cn^\gamma}{dn^\delta}\right), 1 \leqslant n < +\infty\right\}$.

Let $F(x, y)$ be the distribution function of the random vector (U, V) towards which the sequence of random vectors $\{(U_n, V_n), 1 \leqslant n < +\infty\}$ converges pretty surely. For $1 \leqslant n < +\infty$, the random vector $(\overline{Y}_n, \overline{\nu}_n)$ where $\overline{Y}_n = an^\alpha, + bn^\beta U_n, \overline{\nu}_n = cn^\gamma + dn^\delta V_n$, has the same distribution function as the random vector (Y_n, ν_n). For this reason, $\overline{Y}_{\overline{\nu}_n}$ has the same distribution function as Y_{ν_n}, for $1 \leqslant n < +\infty$. Consequently, the survey of the asymptotic distribution of the random process $\{Y_{\nu_n}, 1 \leqslant n < +\infty\}$ is reduced to the survey of the asymptotic distribution of the random process $\{Y_{\overline{\nu}_n}, 1 \leqslant n < +\infty\}$. Obviously, for $n \to +\infty$ we have

$$\overline{Y}_n = an^\alpha + bn^\beta V + O(n^\beta), \quad \overline{\nu}_n = cn^\gamma + dn^\delta V + 0(n^\delta) \tag{2.3}$$

From (2.3) there follows:

$$\begin{aligned} \overline{Y}_{\overline{\nu}_n} = a[cn^\gamma + dn^\delta V + 0(n^\delta)]^\alpha + b[cn^\gamma + dn^\delta V + 0(n^\delta)]^\beta U + \\ + 0\{[cn^\gamma + dn^\gamma V + 0(n^\delta)]^\beta\}. \end{aligned} \tag{2.4}$$

Developing (2.4) in accordance with the powers of n and taking into account only the higher degree terms, we get all the five cases of the theorem.

3. In connection with the asymptotic distribution of the random process of the normalized sums of independent random variables identically distributed, when passing from the ordinary time to the random time, one proves in [3]

Theorem 2. Let $\{X_j, 1 \leqslant j < +\infty\}$ be a sequence of independent random variables, identically distributed, $M(X_j) = 0$, $D^2(X_j) = 1$, $1 \leqslant j < +\infty$, $Y_n = \sum_{j=1}^{n} X_j / \sqrt{n}$, $1 \leqslant n < +\infty$.

If $\{\nu_n, 1 \leqslant n < +\infty\}$ is a sequence of random variables with positive integer values so that the sequence of random variables $\left\{\frac{\nu_n}{n}, 1 \leqslant n < +\infty\right\}$

converges in probability towards a ν positive random variable whatever, then we have:

$$\lim_{n\to+\infty} P(Y_{\nu_n} < x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left(-\frac{t^2}{2}\right) dt. \tag{3.1}$$

Proof. In agreement with Lindeberg's central limit theorem, from the hypothesis of Theorem 2, there follows:

$$\lim_{n\to+\infty} P(Y_n < x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left(-\frac{t^2}{2}\right) dt. \tag{3.2}$$

Applying case V of Theorem 1 for $a = 0$, $b = 1$, $c = 0$, $\beta = 0$, $\delta = 1$ we obtain (3.1).

4. Consider the sequence of real numbers $\{B_n,\ 1 \leqslant n < +\infty\}$ so that $[B_n = n^{\alpha} L(n)$, $(\alpha > 0)$ and $\lim_{n\to+\infty} \frac{L([cn])}{L(n)} = 1$, for any $c > 0$.

In connection with the asymptotic distribution of the random process of the normalized sums independent random variables non-identically distributed, when passing from the ordinary time to the random time, one proves in [4]:

Theorem 3. Let $\{X_j,\ 1 \leqslant j < +\infty\}$ be a sequence of independent random variables, non-identically distributed, $M(X_j) = 0$, $D^2(X_j) < +\infty$, $1 \leqslant j < +\infty$, $Y_n, = \sum_{j=1}^{n} X_j$, $1 \leqslant n < +\infty$ and $B_n^2 = D^2(Y_n)$ where the sequence of real numbers $\{B_n,\ 1 \leqslant n < +\infty\}$ satisfies the previous conditions. Let $\{\nu_n,\ 1 \leqslant n < +\infty\}$ a be sequence of random variables with positive integer values, so that the sequence of random variables $\left\{\frac{\nu_n}{n},\ 1 \leqslant n < +\infty\right\}$ converges in probability towards a positive random variable whatever ν. If

$$\lim_{n\to+\infty} P(Y_n < B_n x) = F(x) \tag{4.1}$$

where $F(x)$ is a continuous distribution function, then we get

$$\lim_{n\to+\infty} P(Y_{\nu_n} < B_n x) = \int_0^{+\infty} F\left(\frac{x}{y\alpha}\right) dH(y) \tag{4.2}$$

where $H(y)$ is the distribution function of the random variable ν.

Proof. In accordance with Corollary 1 of Theorem 4 in [5], we have:

$$\lim_{n\to+\infty} P\left(\frac{Y_n}{B_n} < x,\ \frac{\nu_n}{n} < y\right) = F(x)H(y). \tag{4.3}$$

Applying case V to Theorem 1 for $a = 0$, $c = 0$, $c = 0$, $d = 1$, $\alpha = \beta$, $\delta = 1$ we get (4.2).

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DE LA REPARTITION ASYMTOTIQUE D'UN PROCESSUS ALEATOIRE A TEMPS ALEATOIRE

On étudie le problème suivant: Si $\{X_j, 1 \leqslant j \leqslant \infty\}$ est une suite de variables aléatoires indépendentes et $Y_n = \sum_{j=1}^{n} X_j$; et si $\{\nu_n, 1 \leqslant n < \infty\}$ est une suite de variables aléatoires à valeurs entières positives, on demande de trouver la répartition de la suite $\{Y_{\nu_n}, 1 < n < \infty\}$, c'est-à-dire la répartition de la somme d'un nombre aléatoire de variables aléatoires.

Pour ce faire, l'auteur renonce à l'hypothèse d'indépendance, imposant des restrictions plus faibles, soit à la répartition commune des variables (Y_n, ν_n), soit au genre de conditions d'e godicité de la suite ν_n, $1 \leqslant n < \infty\}$.

ОБ АСИМПТОТИЧЕСКОМ РАСПРЕДЕЛЕНИИ ОДНОГО СЛУЧАЙНОГО ПРОЦЕССА СО СЛУЧАЙНЫМ ВРЕМЕНЕМ

В работе рассматривается следующая задача: Если $\{X_j, 1 \leqslant j < \infty\}$ является рядом случайных независимых переменных, и $Y_n = \sum_{j=1}^{n} X_j$; и если $\{\nu_n, 1 \leqslant n < \infty\}$ является рядом случайных переменных с целыми позитивными значениями, то необходимо найти распределение ряда $\{Y_{\nu_n}, 1 < n < \infty\}$, то есть найти разпределение суммы случайного числа случайных переменных.

С этой целью автор отказывается от допущения независимости, ставя менее строгие ограничения, либо на общее распределение переменных (Y_n, ν_n), либо на тип ограничений по эргодичности относительно ряда $\{\nu_n, 1 \leqslant n < \infty\}$.

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