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The Influence of the Discount Rate on the Optimal Trajectories in a Model of Economic Growth

The paper contains the analysis of a one-sector economic growth model, in which the objective function is the sum of discounted utility over the prognosis period. The utility function depends on two variables: the consumption and the level of fixed funds. The influence of the discount rate on the optimal trajectories is studied.

1. INTRODUCTION

In a previous paper [4], we presented a one sector economic growth model, differing from the usual models by the fact that its utility function is a function of two variables: one of them, as usually, the consumption, and the other, the level of fixed funds. The objective function of the model is the sum of utility over the whole prognosis period.

The purpose of the present paper is the analysis of an analogous growth model, in which the objective function is the sum of discounted utility over the prognosis period. Given the discount rate ρ, the objective function is of the form:

$$V = \sum_{n=0}^{N-1} (1 + \rho)^{-n} U(C_n, K_n).$$

The introduction of the discount rate is justified by the fact that the utility of consumtion of goods and services is greater, when the moment of obtaining the respective goods is closer.

On the other hand, any decision about the economic strategy for a future period assumes the confronting of the present and future interests of the society; the introduction of the discount rate is meant to assure the conciliation of these interests.

2. THE MODEL

We make the following assumptions about the elements of the model:

a) The production function is linear with respect to K and independent of L: $F(K, L) = \beta K$, where K stands for the level of fixed funds and L for the level of labour force.

b) The utility function depends not only on the level of consumption, but on the level of fixed funds as well

$$U(C, K) = C^{1-\alpha}K^{\alpha}; \quad 0 \leqslant \alpha < 1$$

where C denotes the consumption.

c) The objective function is the discounted sum of utility over the whole prognosis horison, with the discount rate ρ:

$$V = \sum_{n=0}^{N-1} (1 + \rho)^{-n} \; U(C_n, K_n)$$

d) The control parameter is the rate of accumulation, u, whose range is the closed interval $[\gamma, \delta]$, $0 \leqslant \gamma < \delta \leqslant 1$.

With these specifications, we obtain the following one-sector model:

$$\begin{cases} K_{n+1} = (1 - \mu + u_n\beta)\, K_n \quad n = 0, 1, \ldots, N-1; \quad K_0 > 0 \\ u_n \in [\gamma, \delta] \\ max \sum_{n=0}^{N-1} (1 + \rho)^{-n}\, C_n^{1-\alpha} K_n^{\alpha} \end{cases}$$

where μ is the coefficient of physical and moral wear and tear of fixed funds.

Remark

Since C depends on K according to relation $C_n = (1 - u_n)\,\beta K_n$ the performance of the model can be written as follows:

$$\sum_{n=0}^{N-1} (1 + \rho)^{-n}\, K_n [(1 - u_n)\, \beta]^{1-\alpha}.$$

3. NECESSARY OPTIMALITY CONDITIONS *

The above model is a special case of a control system, in which neither isoperimetric conditions, nor final condition upon the state variable K appear.

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* A concise description of a general control system and the statement of the necessary optimality conditions can be found in [4].

The Hamiltonian function of the system is:

$$H_n(K, u, \tilde{\psi}_n) = \tilde{\psi}_n(1 - \mu + \beta u)\, K + \lambda_0(1 + \rho)^{-n}\, U[(1 - u)\, \beta K, K]$$

where $\tilde{\psi}_n$ is the solution of the dual equation **

$$\psi_{n-1} = \psi_n(1 - \mu + \beta \tilde{u}_n) +$$

$$+ \lambda_0 \left[\frac{\partial U}{\partial x} (\tilde{C}_n, \tilde{K}_n) \cdot (1 - \tilde{u}_n)\, \beta + \frac{\partial U}{\partial y} (\tilde{C}_n, \tilde{K}_n) \right] (1 + \rho)^{-n}$$

with final condition $\tilde{\psi}_{N-1} = \lambda_1$.

According to theorem 2.1 from [4], the multipliers λ_0 and λ_1 satisfy the following conditions: $\lambda_0 \geqslant 0$, $\lambda_1 \geqslant 0$; $\lambda_0 + \lambda_1 > 0$.

The absence of any final condition upon the variable K implies $\lambda_1 = 0$ and hence $\lambda_0 > 0$; without loss of generality, we may assume $\lambda_0 = 1$; then, the Hamiltonian function becomes:

$$H_n(K, u, \tilde{\psi}_n) = K\{\tilde{\psi}_n(1 - \mu + \beta u) + (1 + \rho)^{-n}[\beta(1 - u)]^{1-\alpha}\}.$$

The adjoint equation can be written:

a) $\tilde{\psi}_{n-1} = \tilde{\psi}_n(1 - \mu + \beta \tilde{u}_n) + (1 + \rho)^{-n}[\beta(1 - \tilde{u}_n)]^{1-\alpha}$.

The necessary optimality conditions are:

b) $$\frac{\partial H_n}{\partial u} (\tilde{K}_n, \tilde{u}_n, \tilde{\psi}_n)\, (u - \tilde{u}_n) = \beta \tilde{K}_n \left[\tilde{\psi}_n - \frac{\partial U}{\partial x} (\tilde{C}_n, \tilde{K}_n)\, (1 + \rho)^{-n} \right] (u - \tilde{u}_n) \leqslant 0$$

c) $\tilde{\psi}_{N-1} = 0$.

4. THE SUFFICIENCY OF THE NECESSARY OPTIMALITY CONDITIONS

The same as for the MUBELIN model, we can state the following theorem:

Theorem 1

The necessary optimality conditions are also sufficient optimality conditions. The proof is analogous to that given in [4] (Theorem 3.1); the modified form of the objective function does not alter the main points of the argument.

** We noted by $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ the partial derivatives of the function $U(.,.)$ with respect to the first and the second variable, respectively.

Considering the optimum condition b) explicitely, we distinguish the following cases:

b.1) $\tilde{u}_n = \gamma \Leftrightarrow \tilde{\psi}_n \leqslant (1-\alpha)(1+\rho)^{-n}[\beta(1-\gamma])^{-\alpha}$

b.2) $\tilde{u}_n = \delta \Leftrightarrow \tilde{\psi}_n \geqslant (1-\alpha)(1+\rho)^{-n}[\beta(1-\delta)]^{-\alpha}$

b.3) $\tilde{u}_n \in (\gamma, \delta) \Leftrightarrow \tilde{\psi}_n = (1-\alpha)(1+\rho)^{-n}[\beta(1-\tilde{u}_n)]^{-\alpha}$.

The corresponding forms for the adjoint equation are:

a.1) $\tilde{\psi}_{n-1} = \tilde{\psi}_n(1-\mu+\beta\gamma) + (1+\rho)^{-n}[\beta(1-\gamma)]^{1-\alpha}$

a.2) $\tilde{\psi}_{n-1} = \tilde{\psi}_n(1-\mu+\beta\delta) + (1+\rho)^{-n}[\beta(1-\delta)]^{1-\alpha}$

a.3) $\tilde{\psi}_{n-1} = \tilde{\psi}_n\left(1-\mu+\frac{\beta}{1-\alpha}-\frac{\alpha\beta\tilde{u}_n}{1-\alpha}\right)$.

Since from the final condition c) we have

$$\tilde{\psi}_{N-1} = 0 < (1-\alpha)(1+\rho)^{1-N}[\beta(1-\gamma)]^{-\alpha}$$

it follows that $\tilde{u}_{N-1} = \gamma$.

Supposing that $\tilde{u}_{N-2} = \ldots = \tilde{u}_{N-m-1} = \gamma$, and solving the difference equation a.1), we obtain

i) $\tilde{\psi}_{N-p-1} = (1+\rho)^{-(N-1)}[\beta(1-\gamma)]^{1-\alpha}\dfrac{(1-\mu+\beta\gamma)^p-(1+\rho)^p}{\beta\gamma-(\mu+\rho)}$

$p = 0, 1, \ldots, m$, if $\beta\gamma \neq \mu+\rho$.

ii) $\tilde{\psi}_{N-p-1} = p(1+\rho)^{-(N-p)}[\beta(1-\gamma)]^{1-\alpha}$

$p = 0, 1, \ldots, m$, if $\beta\gamma = \mu+\rho$.

Since $\tilde{\psi}_{N-p-1}$ increases as p increases, it may happen that, for a given p, the inequality b.1) is no longer valid.

Let m be the last index for which this inequality is valid, i.e.

$$\tilde{\psi}_{N-m-1} \leqslant (1-\alpha)(1+\rho)^{-(N-m-1)}[\beta(1-\gamma)]^{-\alpha}$$

$$\tilde{\psi}_{N-m-2} > (1-\alpha)(1+\rho)^{-(N-m-2)}[\beta(1-\gamma)]^{-\alpha}.$$

We have

i) $m = \left[\dfrac{\ln\{(1-\alpha)(\beta\gamma-\mu-\rho)+\beta(1-\gamma)\}-\ln\beta-\ln(1-\gamma)}{\ln(1-\mu+\beta\gamma)-\ln(1+\rho)}\right]$,

if $\beta\gamma \neq \mu+\delta$, where $[x]$ stands for the integer part of number x; and

ii) $m = \left[\dfrac{(1-\alpha)(1+\rho)}{\beta(1-\gamma)}\right]$,

if $\beta\gamma = \mu+\rho$.

Since we have $\tilde{u}_{N-m-2} \neq \gamma$, there exist two possibilities:

1) $(1-\alpha)(1+\delta)^{-(N-m-2)}[\beta(1-\gamma)]^{-\alpha} < \tilde{\psi}_{N-m-2} <$

$< (1-\alpha)(1+\rho)^{-(N-m-2)}[\beta(1-\delta)]^{-\alpha}$

and then $\tilde{u}_{N-m-2} \in (\gamma, \delta)$

2) $\tilde{\psi}_{N-m-2} \geq (1-\alpha)(1+\rho)^{-(N-m-2)}[\beta(1-\delta)]^{-\alpha}$

and then $\tilde{u}_{N-m-2} = \delta$.

We shall prove some propositions which characterize the behaviour of the optimal solution.

Proposition 1

If $\beta > \mu$, then $\tilde{u}_n \in (\gamma, \delta)$ implies $\tilde{\psi}_{n-1} > \tilde{\psi}_n$.

Proof

If $\tilde{u}_n \in (\gamma, \delta)$, then we have for $\tilde{\psi}_{n-1}$ the equation a.3):

$$\tilde{\psi}_{n-1} = \tilde{\psi}_n \left(1 - \mu + \frac{\beta}{1-\alpha} - \frac{\alpha\beta\tilde{u}_n}{1-\alpha}\right).$$

In order to prove that $\tilde{\psi}_{n-1} > \tilde{\psi}_n$, it is sufficient to show that

$$\frac{\beta}{1-\alpha}(1 - \alpha\tilde{u}_n) - \mu > 0.$$

Indeed,

$$\frac{\beta(1-\alpha\tilde{u}_n)}{1-\alpha} - \mu > \frac{\beta(1-\alpha\delta)}{1-\alpha} - \mu > \beta - \mu > 0.$$

Proposition 2

If $\beta > \mu$, then $\tilde{u}_n = \delta$ implies $\tilde{u}_{n-1} = \delta$. If, moreover, $\beta\delta > \mu$, then $\tilde{u}_n = \delta$ implies $\tilde{\psi}_{n-1} > \tilde{\psi}_n$.

Proof

If $\tilde{u}_n = \delta$, then $\tilde{\psi}_n \geq (1-\alpha)(1+\rho)^{-n}[\beta(1-\delta)]^{-\alpha}$, and, from a.2), $\tilde{\psi}_{n-1} = \tilde{\psi}_n(1 - \mu + \beta\delta) + (1+\rho)^{-n}[\beta(1-\delta)]^{-\alpha}$.

Hence

$$\tilde{\psi}_{n-1} \geq (1-\alpha)(1+\rho)^{-n}[\beta(1-\delta)]^{-\alpha}\left[1 - \mu + \beta\delta + \frac{\beta(1-\delta)}{1-\alpha}\right] >$$

$$> (1-\alpha)(1+\rho)^{-n}[\beta(1-\delta)]^{-\alpha}.$$

It follows $\tilde{u}_{n-1} = \delta$.

The inequality $1 - \mu + \beta\delta + \frac{\beta(1-\delta)}{1-\alpha} > 1$ is valid, since it reduces to the obvious inequality $\beta \frac{1-\alpha\delta}{1-\alpha} > \mu$.

If $\beta\delta > \mu$, then it obviously follows from a.2) that $\tilde{\psi}_{n-1} \geq \tilde{\psi}_n$.

Proposition 3

If $\tilde{u}_n, \tilde{u}_{n+1} \in (\gamma, \delta)$ and $\rho < \frac{\beta(1-\alpha\delta)}{1-\alpha} - \mu$, then $\tilde{\psi}_n > \tilde{\psi}_{n+1}(1+\rho)$.

Proof

From the adjoint equation a), we have

$$\tilde{\psi}_n = \tilde{\psi}_{n+1}(1 - \mu + \beta\tilde{u}_{n+1}) + (1+\rho)^{-(n+1)} [\beta(1 - u_{n+1})]^{1-\alpha}.$$

From b.3) it follows

$$\tilde{\psi}_{n+1} = (1-\alpha)(1+\rho)^{-(n+1)} [\beta(1 - u_{n+1})]^{-\alpha}.$$

Hence

$$\frac{\tilde{\psi}_n}{\tilde{\psi}_{n+1}} = \frac{(1+\rho)^{-(n+1)} [\beta(1-\tilde{u}_{n+1})]^{-\alpha} [(1-\alpha)(1-\mu+\beta\tilde{u}_{n+1}) + \beta(1-\tilde{u}_{n+1})]}{(1-\alpha)(1+\rho)^{-(n+1)} [\beta(1-u_{n+1})]^{-\alpha}}$$

or

$$\frac{\tilde{\psi}_n}{\tilde{\psi}_{n+1}} = \frac{(1-\alpha)(1-\mu) + \beta(1-\alpha\tilde{u}_{n+1})}{1-\alpha} > 1 - \mu + \frac{\beta(1-\alpha\delta)}{1-\alpha} > 1 + \rho.$$

Remark

The condition $\rho < \frac{\beta(1-\alpha\delta)}{1-\alpha} - \mu$ is plausible from the economic point of view, since to ρ are given, as a rule, values less than 0.15, the value of β is about 0.3, the coefficient μ is about 0.05 and the ratio $\frac{1-\alpha\delta}{1-\alpha}$ is greater than 1.

Theorem 2

Assuming $\beta > \mu$, in the behaviour of the optimal control for the model with discount three phases can be distinguished, namely:

Phase 1) $\tilde{u}_n = \delta$, $\quad n = 0, 1, \ldots, n_1^*$

Phase 2) $\tilde{u}_n \in (\gamma, \delta)$, $\quad n = n_1^* + 1, \ldots, n_2^*$

Phase 3) $\tilde{u}_n = \gamma$, $\quad n = n_2^* + 1, \ldots, N - 1$.

According to the values of the parameters of the model, one of the first two phases or both of them can be absent.

The theorem is a direct consequence from propositions 1 and 2.

Theorem 3

Under the conditions of proposition 3, the optimal control $\tilde{u}_n$ is a non-increasing function of n.

Proof

If phases 1) and 2) are absent, then $\tilde{u}_n$ is constant, equal to γ. If only phase 2) is absent, then the optimal control is piece-wisely constant, having only one switch from $\tilde{u}_{n^*} = \delta$ to $\tilde{u}_{n^*+1} = \gamma$.

In the general case, the situation is as follows. At the moment n_1^* of the first switch, the optimal control decreases from $\tilde{u}_{n_1^*} = \delta$ to $\tilde{u}_{n_1^*+1} \in (\gamma, \delta)$. At the moment n_2^* of the second switch, the optimal control decreases from $\tilde{u}_{n_2^*} \in (\gamma, \delta)$ to $\tilde{u}_{n_{2+1}^*} = \gamma$.

It remains to be shown that in phase 2) $\tilde{u}_n$ is decreasing. Let n and $n+1$ be two consecutive moments in phase 2). Then, from the optimum condition b.3), we have

$$\tilde{\psi}_n = (1-\alpha)(1+\rho)^{-n}[\beta(1-\tilde{u}_n)]^{-\alpha}$$

$$\tilde{\psi}_{n+1} = (1-\alpha)(1+\rho)^{-(n+1)}[\beta(1-\tilde{u}_{n+1})]^{-\alpha}$$

and hence

$$\tilde{u}_n = 1 - \frac{1}{\beta}\left[\frac{1-\alpha}{\tilde{\psi}_n}\right]^{1/\alpha}(1+\rho)^{-n/\alpha}$$

$$\tilde{u}_{n+1} = 1 - \frac{1}{\beta}\left[\frac{1-\alpha}{\tilde{\psi}_{n+1}}\right]^{1/\alpha}(1+\rho)^{-(n+1)/\alpha}.$$

From proposition 3, we have $\tilde{\psi}_n > \tilde{\psi}_{n+1}(1+\rho)$, whence $\tilde{u}_{n+1} < \tilde{u}_n$.

6. COMPARISON BETWEEN THE "MUBELIN" MODEL AND THE MODEL WITH DISCOUNT

One of the aims of this paper is the study of the changes in the behaviour of the optimal control, caused by the introduction of the discount factor in the initial MUBELIN model.

In particular, we shall refer to the changes in the value of m ($m+1$ is the number of moments when $\tilde{u}_n = \gamma$).

Let us note by m_1 and m_2 the values of m corresponding to the case $\rho = 0$ and $\rho \neq 0$, respectively (considering the rest of parameters the same in both models).

Let us note by ψ_n^1, ψ_n^2 the dual variables corresponding to the MUBELIN model and the model with discount, respectively.

We remark that from inequalities b.1) and i) (section 5), it follows

$$\beta(1-\gamma)\frac{\left(\frac{1-\mu+\beta\gamma}{1+\rho}\right)^{p}-1}{\beta\gamma-(\mu+\rho)}\leqslant 1-\alpha,$$

and from inequality

$$\psi^{1}_{N-p-1}\leqslant\frac{\partial U(C_{N-p-1},\,K_{N-p-1})}{\partial x}$$

it follow

$$\beta(1-\gamma)\frac{(1-\mu+\beta\gamma)^{p}-1}{\beta\gamma-\mu}\leqslant 1-\alpha.$$

It is natural, then to compare the following expressions:

$$f^{1}(p)=\frac{(1-\mu+\beta\gamma)^{p}-1}{\beta\gamma-\mu},\quad f^{2}(p)=\frac{\left(\frac{1-\mu+\beta\gamma}{1+\rho}\right)^{p}-1}{\beta\gamma-(\mu+\rho)}.$$

Since both expressions are increasing functions of p, this comparison will give immediately the relation between the values m_1 and m_2.

According to the values of the parameters, three cases are distinguished, namely:

Case 1): $\beta\gamma>\mu+\rho$.

Comparing the expressions $f^{1}(p)$ and $f^{2}(p)$, we obtain, by some simple computations, the inequality

$$\frac{1-(1+\rho)^{-p}}{1-(1-\mu+\beta\gamma)^{-p}}\leqslant\frac{\rho}{\beta\gamma-\mu}<1$$

which implies $f^{1}(p)\leqslant f^{2}(p)$; hence, $m_2\leqslant m_1$.

Case 2): $\rho>\beta\gamma-\mu>0$.

Comparing $f^{1}(p)$ and $f^{2}(p)$, we obtain the inequality

$$\frac{1-(1+\rho)^{-p}}{1-(1+\mu+\beta\gamma)^{-p}}\geqslant\frac{\rho}{\beta\gamma-\mu}>1$$

which implies $f^{1}(p)\leqslant f^{2}(p)$; it follows, as in case 1), $m_2\leqslant m_1$.

Case 3): $\rho>0>\beta\gamma-\mu$.

Assuming that the inequality

(I) $$\rho+\mu-\beta\gamma\leqslant\rho(1-\mu+\beta\gamma)^{-p}+(\mu-\beta\gamma)(1+\rho)^{-p}$$

holds, we obtain $f^{2}(p)\leqslant f^{1}(p)$ and hence $m_1\leqslant m_2$.

In order to prove inequality (I), let us note $\mu-\beta\gamma=b$. We have $b>0$; moreover, economically plausible assumptions assure that $b<1$.

Inequality (I) becomes: $\rho+b\leqslant\rho(1-b)^{-p}+b(1+\rho)^{-p}$.

Let us consider the function of p

$$h(p) = \frac{\rho}{(1-b)^p} + \frac{b}{(1+\rho)^p};$$

we have $h(0) = \rho + b$ and $h(1) = \frac{\rho}{1-b} + \frac{b}{1+\rho} > \rho + b$.

Making use of the theorem of finite increments on the interval [0, 1], we get the existence of a value $p_0 \in (0, 1)$ such that $h(1) - h(0) = h'(p_0) > 0$. On the other hand, we have $h''(p) > 0$ for any p, hence h' is an increasing function of p; consequently, $h'(p) > 0$ for any $p \geq p_0$. Since $p_0 \in (0, 1)$, it follows that inequality (I) holds for any $p \in N$.

7. CONCLUSIONS DEDUCED FROM EXPERIMENTAL COMPUTATIONS

On the basis of the qualitative properties deduced in section 5, an algorithm for the computation of the optimal solution was worked out, as well as a programme which was tested for three values of ρ: 0.0; 0.08; 0.15. For every value of ρ, ten variants were tested, with α taking values from 0.0 to 0.9 (with a step of 0.1). The other parameters of the model were preserved constant in all the variants, assigned with the following values:

$$\beta = 0.33; \quad \gamma = 0.1; \quad \delta = 0.4; \quad \mu = 0.05; \quad K_0 = 1{,}000.$$

In the following tables, the optimum values of the control and of the main output-variables are given, for all the tested variants.

In table IV the values of the optimal controls for all the tested values of the parameter ρ are given.

From the comparative analysis of these results, we can derive the following conclusions about the variations of the optimal solution with respect to the values of the parameter ρ:

i) for $\alpha = 0.9$ and 0.7, the optimal solution is the same for the three tested values of ρ. Moreover, the optimal solutions for $\alpha = 0.6$, 0.4 and 0.1 coincide in the cases $\rho = 0$ and $\rho = 0.08$.

ii) as the value of ρ increases, the number of "intermediate" values (different from γ and δ) of the optimal control increases also. Thus, for $\rho = 0$ we have only one intermediate value, in the case $\alpha = 0.5$. For $\rho = 0.08$, there are three intermediate values, in the cases $\alpha = 0.8$; 0.5; 0.3. For $\rho = 0.15$, there are four intermediate values, in the cases $\alpha = 0.8$; 0.6; 0.4; 0.2.

iii) as follows also from the theoretical arguments of section 6, in the case $\beta\gamma - \mu < 0$ (the experimental data satisfy this inequality) the number of moments when the optimal control takes the value γ remains constant or increases as ρ increases. Thus, in the variants with $\alpha = 0.9, \ldots, 0.4$, the number of moments n for which $\tilde{u}_n = \gamma$ is the same for the three values of ρ considered.

For $\alpha = 0.3$, in the cases $\rho = 0$ and $\rho = 0.08$, we have $\tilde{u}_n = \gamma$ for three values of n and in the case $\rho = 0.15$, $\tilde{u}_n = \gamma$ for four values of n.

For $\alpha = 0.2$, in the case $\rho = 0$, we have $\tilde{u}_n = \gamma$ for three values of n and in the other two cases $\tilde{u}_n = \gamma$ for four values of n.

For $\alpha = 0.1$, in the cases $\rho = 0$ and $\rho = 0.08$ we have $\tilde{u}_n = \gamma$ for four values of n and in the case $\rho = 0.15$, $\tilde{u}_n = \gamma$ for five values of n.

Eventually, for $\alpha = 0$, we have the following situation; in the case $\rho = 0$, $\tilde{u}_n = \gamma$ for four values of n; in the case $\rho = 0.08$, $\tilde{u}_n = \gamma$ for five values of n, and in the case $\rho = 0.15$, $\tilde{u}_n = \gamma$ for six values of n.

In conclusion, the introduction of the discount factor has as consequence a more realistic behaviour of the optimal trajectories.

Table I

$\rho = 0.0$

Variant	α	Optimal control	C_0	C_{29}	K_{30}	Performance
I.1	0.9	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	198	2919	9663	100297
I.2	0.8	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	198	2919	9663	85607
I.3	0.7	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	198	2652	8779	73177
I.4	0.6	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	198	2652	8779	62677
I.5	0.5	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.351;\ u_{28} = u_{29} = 0.1$	198	2613	8649	53698
I.6	0.4	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = u_{28} = u_{29} = 0.1$	198	2409	7976	46092
I.7	0.3	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = u_{28} = u_{29} = 0.1$	198	2409	7976	39621
I.8	0.2	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = u_{28} = u_{29} = 0.1$	198	2409	7976	34070
I.9	0.1	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = u_{27} = u_{28} = u_{29} = 0.1$	198	2189	7246	29342
I.10	0.0	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = u_{27} = u_{28} = u_{29} = 0.1$	198	2189	7246	25312

Table II

$\rho = 0.08$

Variant	α	Optimal control	C_0	C_{29}	K_{30}	Performance
II.1	0.9	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	198	2919	9663	26248
II.2	0.8	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = 0.396,\ u_{29} = 0.1$	198	2915	9650	22357
II.3	0.7	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	198	2652	8779	19060
II.4	0.6	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	198	2652	8779	16259
II.5	0.5	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.185;\ u_{28} = u_{29} = 0.1$	198	2478	8202	13877
II.6	0.4	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = u_{28} = u_{29} = 0.1$	198	2409	7976	11857
II.7	0.3	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = 0.295;\ u_{27} = u_{28} =$ $= u_{29} = 0.1$	198	2332	7720	10134
II.8	0.2	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = u_{27} = u_{28} = u_{29} = 0.1$	198	2189	7246	8671
II.9	0.1	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = u_{27} = u_{28} = u_{29} = 0.1$	198	2189	7246	7424
II.10	0.0	$u_n = 0.4,\ n = 0, \ldots, 24$ $u_n = 0.1;\ n = 25, \ldots, 29$	198	1989	6583	6361

Table III

$\rho = 0.15$

Variant	α	Optimal control	C_0	C_{29}	K_{30}	Performance
III.1	0.9	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	198	2919	9663	12078
III.2	0.8	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = 0.346;\ u_{29} = 0.1$	198	2871	9505	10278
III.3	0.7	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	198	2652	8779	8749
III.4	0.6	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.334;\ u_{28} = u_{29} = 0.1$	198	2598	8601	7449
III.5	0.5	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = u_{28} = u_{29} = 0.1$	198	2409	7976	6345
III.6	0.4	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_{26} = 0.311;\ u_n = 0.1,$ $n = 27, 28, 29$	198	2344	7759	5406
III.7	0.3	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$	198	2189	7246	4608
III.8	0.2	$u_n = 0.4,\ n = 0, \ldots, 24$ $u_{25} = 0.247;\ u_n = 0.1;$ $n = 26, \ldots, 29$	198	2087	6907	3929
III.9	0.1	$u_n = 0.4,\ n = 0, \ldots, 24$ $u_n = 0.1;\ n = 25, \ldots, 29$	198	1989	6583	3352
III.10	0.0	$u_n = 0.4;\ n = 0, \ldots, 23$ $u_n = 0.1;\ n = 24, \ldots, 29$	198	1807	5981	2862

Table IV

Variant	α	Optimal control $\rho = 0.0$	Optimal control $\rho = 0.08$	Optimal control $\rho = 0.15$
1	0.9	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$
2	0.8	$u_n = 0.4,\ n = 0, \ldots, 28$ $u_{29} = 0.1$	$u_n = 0.4, n = 0, \ldots, 27$ $u_{28} = 0.396;\ u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = 0.346;\ u_{29} = 0.1$
3	0.7	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	$u = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$
4	0.6	$u_n = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	$u = 0.4,\ n = 0, \ldots, 27$ $u_{28} = u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.334; u_{28} = u_{29} = 0.1$
5	0.5	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.351;\ u_{28} = u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_{27} = 0.185;\ u_{28} = u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_n = 0.1;\ n = 27, 28, 29$
6	0.4	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_n = 0.1;\ n = 27, 28, 29$	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_n = 0.1;\ n = 27, 28, 29$	$u = 0.4, n = 0, \ldots, 25$ $u_n = 0.1;\ n = 27, 28, 29$ $u_{26} = 0.311$
7	0.3	$u_n = 0.4,\ n = 0, \ldots, 26$ $u_n = 0.1;\ n = 27, 28, 29$	$u_n = 0.4; n = 0, \ldots, 25$ $u_{26} = 0.295;\ u_{27} = u_{28} = u_{29} = 0.1$	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$
8	0.2	$u_n = 0.4, n = 0, \ldots, 26$ $u_n = 0.1;\ n = 27, 28, 29$	$u_n = 0.4;\ n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$	$u_n = 0.4, n = 0, \ldots, 24$ $u_{25} = 0.247;\ u_n = 0.1;$ $n = 26, \ldots, 29$
9	0.1	$u_n = 0.4, n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$	$u_n = 0.4, n = 0, \ldots, 24$ $u_n = 0.1;\ n = 25, \ldots, 29$
10	0.0	$u_n = 0.4,\ n = 0, \ldots, 25$ $u_n = 0.1;\ n = 26, \ldots, 29$	$u_n = 0.4,\ n = 0, \ldots, 24$ $u_n = 0.1;\ n = 25, \ldots, 29$	$u_n = 0.4, n = 0, \ldots, 23$ $u_n = 0.1;\ n = 24, \ldots, 29$

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L'ETUDE DE L'INFLUENCE DU COEFFICIENT DE MISE A JOUR SUR LES TRAJECTOIRES OPTIMALES DANS UN MODELE DE CROISSANCE ECONOMIQUE

Les auteurs étudient l'influence du coefficient de mise à jour sur les trajectoires optimales dans un modèle unisectoriel de croissance économique, à variables discrètes et à fonction d'utilité dépendant explicitement du niveau des fonds fixes.

ИССЛЕДОВАНИЕ ВЛИЯНИЯ НОРМЫ ДИСКОНТ-ФАКТОРА НА ОПТИМАЛЬНЫЕ ТРАЕКТОРИИ В МОДЕЛИ ЭКОНОМИЧЕСКОГО РОСТА

Авторы рассматривают влияние нормы дисконт-фактора на оптимальные траектории в односекторной модели экономического роста, с дискретными переменными и с функцией полезности, явно зависящей от уровня основных фондов.

3.VIII.1977