C. V. NEGOIȚĂ
M. SULARIA
Institute of Management
and Informatics
Bucharest, Romania

A Selection Method of Nondominated Points in Multicriteria Decision Problems

This paper presents a new algorithm in order to select a maximal element of some subsets of R^n. The selection is based on the distance between the maximal element and the ideal point. This distance is a fuzzy subset. The algorithm has a finite number of steps.

1. INTRODUCTION

The increasing importance of planning is one of the interesting aspects of the world today. Any planning problem can be characterized, at least, by two requirements. There should be a number of objectives which has to be fulfilled simultaneously and the decision variables must satisfy of side conditions. These requirements determine the basic structure of the planning problem.

In the last years, important studies have been made to effectively use mathematical programming as part of real planning processes.

There has been considerable recent interest in planning under inexactness, which can be usefully stated as planning in fuzzy environment.

The present paper may be seen as the outgrowth of a whole trend in recent years to correct a realistic decision theory. Rather than enumerate these contributions, we note that the ones that most influenced us have been bound within the covers of [1].

Very recently it has become clear that a fuzzy programming problem can be recast as standard programming from which is solvable by standard algorithms [2], and that, in fact, a fuzzy programming problem is a multiobjective optimization problem [3].

The purpose of this paper is to show that it is possible to make very effective use of non-linear programming results in conjunction with the dis-

placed ideal ideas, such as those in [4] in the construction of efficient optimization algorithms.

It is to be hoped that these recent steps in the direction of merging fuzzy set technology and mathematical programming ideas will contribute to a new generation of efficient planning algorithms.

The concept of a maximal element in a partially ordered set can be used to give a sense to what is called multiobjective optimization.

Consider $\Omega \subseteq \mathbf{R}^n$, $\Omega \neq \emptyset$ and $f: \Omega \to \mathbf{R}^p$ a function, $f = (f_1, f_2, ..., f_p)$, $p \geqslant 1$. An element $x^0 \in \Omega$ is called the point of the function f if $f(x^0)$ is a maximal element in the set $(f(\Omega), \leqslant)$ partially ordered by the product order in $\mathbf{R}^p$, induced by the usual order relation in $\mathbf{R}$. In other words, $x^0 \in \Omega$ is a maximum Pareto point for f if from $x \in \Omega$ and $f(x) \geqslant f(x^0)$ it follows $f(x) = f(x^0)$.

The notion of nondominated points can be exploited to solve the requirements for the exact solution of the multiobjective optimization problem. Theoretical considerations of this approach are discussed in [5].

Consider a subset $\Omega \leqslant \mathbf{R}^n$, $\Omega \neq \emptyset$ which is convex and compact. Generally, the set of maximal points of Ω is rich. In the following we shall give a solution procedure for a maximal point, assuming a point $x^0 \in \Omega$ which is meant as a feasible solution.

Let $x_i^* = \sup_{x \in \Omega} pr_i(x) \quad i = 1, ..., n$

where $pr_i: \mathbf{R}^n \to \mathbf{R}$ is the canonic projection $pr_i(x) = x_i \forall x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$.

The vector $x^* = (x_1^*, x_2^*, ..., x_n^*) \in \mathbf{R}^n$ can be defined as the ideal point. Generally, $x^* \notin \Omega$. The optimization problem:

(P) Determine $x^1 \in \Omega$ with the properties:

(1) $x^1 \geqslant x^0$

(2) x^1 as close as possible to the ideal point x^*

(3) x^1 is a maximal point in Ω.

Define $\Omega_x = \{y \in \mathbf{R}^n \mid y \geqslant x\} \cap \Omega$, $x \in \mathbf{R}^n$.

The degree of closeness of an x^0 to x^* is designated by the function

$$d_{x^*}^{x^\circ}: \Omega_{x^\circ} \to [0, 1]$$

$$d_{x^*}^{x^\circ}(x) = \min_{i \in I_{x^\circ}} \frac{x_i - x_i^\circ}{x_i^* - x_i^\circ}$$

where $x = (x_1, x_2, ..., x_4) \in \Omega_{x^\circ}$.

$$I_{x^\circ} = \{i \in \mathbf{N} / \, 1 \leqslant i \leqslant n, \quad x_i^\circ < x_i^*\}.$$

We shall say that x is closer to x^* than y, if

$$d_{x^*}^{x^\circ}(x) \geqslant d_{x^*}^{x^\circ} \quad (y).$$

Proposition 1: If the domain $\Omega \leqslant \mathbf{R}^n$ is strictly convex, $I_{x^\circ} = \{1, 2, ..., n\}$ and $x^1 \in \Omega$ is a point with the property

$$d_{x^*}^{x^\circ}(x^1) = \sup_{x \in \Omega} d_{x^*}^{x^\circ}(x) \tag{1.1}$$

then x^1 is a nondominated point of Ω and $x^1 \geqslant x^\circ$.

Proof: It can be shown that $x^1 \in \delta \cdot \Omega$ ($\delta\Omega$ = the frontier of Ω), for x^1 which verifies (1.1).

If $x^1 \in \text{Int } \Omega$, then there is a $r > 0$, such that $S(x^1, r) \leqslant \Omega$ where $S(x^1, r) = \{x \in \mathbf{R}^n \mid \| x - x^1 \| < r\}$ is the sphere centered on x^1.

$$\text{Let be } 0 < \varepsilon < \frac{r}{\|x^* - x^0\|} \cdot \text{Then, it follows that}$$

$$\| x^1 + \varepsilon(x^* - x^0) - x^1 \| = \varepsilon \| x^* - x^0 \| < r,$$

$$x = x^1 + \varepsilon(x^* - x^0) \in S(x^1, r)$$

$$x^1 + \varepsilon(x^* - x^0) \in \Omega.$$

Noting also that $x_i = x_i^1 + \varepsilon\ (x_i^* - x_i^\circ) > x_i^1\ \forall_i = 1, ..., n$

$$d_{x^*}^{x^\circ}(x) = \min_{i=1,...,n} \frac{x_i - x_i^\circ}{x_i^* - x_i^\circ} > \min_{i=1,...,n} \frac{x_i^1 - x_i^\circ}{x_i^1 - x_i^0} = d_{x^*}^{x^\circ}(x^1)$$

and the last statement contradicts (1, 1). The results then follows that $x^1 \in \delta\Omega$. This property, bearing in mind that Ω is strictly convex, implies that x^1 is unique. Assume $x^2 \leftarrow \Omega$, verifying (1, 1), and $x^1 \neq x^2$. It is easily shown that $d_{x^*}^{x^\circ}$ is a convex function. Thus, $\forall\ \lambda \in (0, 1)$, since $x^1, x^1 \in \delta\ \Omega$ and Ω is strictly convex.

$$d_{x^*}^{x^\circ}(\lambda x^1 + (1 - \lambda)x^2) \geqslant d_{x^*}^{x^\circ}(x^1) + (1 - \lambda)\ d_{x^*}^{x^\circ}(x^2) = d_{x^*}^{x^\circ}(x^1)$$

and $\lambda x^1 + (1 - \lambda)x^1 \in Int\ \Omega$ would have the properties (1,1) which contradict the fact that the solutions of (1,1) are on the frontier of Ω.

We are ready now to show that x^1 is a nondominated point in Ω.

Let $x \in \Omega$, $x \geqslant x^1$. The, $d_{x^*}^{x^\circ}(x) \geqslant d_{x^*}^{x^\circ}(x^1)$. Since x^1 verifies (1. 1) then x verifies also (1.1), and bearing in mind theat (1.1) has a unique solution, then it follows $x = x^1$. Therefore, x^1 is a nondominated point in Ω.

From (1.1) $d_{x^*}^{x^\circ}(x^1) \geqslant d_{x^*}^{x^\circ}(x^\circ) = 0$,

$$\min \frac{x_i^1 - x_i^\circ}{x_i^* - x_i^\circ} \geqslant 0$$

therefore, $\forall i = i, ..., n$, we have $x_i^1 \geqslant x_i^\circ$, or $x^1 \geqslant x^0$.

Remark 1: If $\Omega \subseteq \mathbf{R}^n$, $x^\circ \in \Omega$, and $\exists i \in \{1, 2, ..., n\}$ such that $i \notin I_{x^\circ}$, then x^0 is a nondominated point in Ω, when Ω is strictly convex, the solution of $x_i^* = \sup_{x \in \Omega} pr_i(x)$, $i = (1, ..., n)$ is unique.

Remark 2: If $I_{x^\circ} = \emptyset$, then $x^0 = x^* \in \Omega$ is a nondominated point in Ω.

Remark 3: If Ω is only convex, proposition 1 is not generally true.

Remark 4: If $x^1 \in \Omega$ has property (1.1) and Ω is compact and nonempty, then $x \in \Omega$, $x \geqslant x^1$ implies the existence of a $i \in \{1, 2, ..., n\}$ with the property $x_i = x_i^1$.

The last remark and proposition (1.1) suggest a selection method for a nondominated point, also for the general case when Ω is convex, compact and nonempty.

Let be $x^0 \in \Omega$.

The procedure has the following steps:

(a) Determine the ideal point x^* in Ω_{x°,

i.e. solve the problems:

$$x_i^* = \sup_{x \in \Omega_{x^\circ}} pr_i(x) \qquad i = 1, \ldots, n;$$

(b) Determine the subset I_{x°. If $I_{x^\circ} = \emptyset$ stop;

otherwise go to (c);

(c) Solve the problem:

$\sup d^{x^\circ}_{x^*}(x) = d^{x^\circ}_{x^*}(x^1)$

and find $x^1 \in \Omega$, its solution;

(d) Determine the ideal point x_1^* of Ω_{x^1};

(e) $x^\circ \to x^1$, $\quad x^* \to x_1^*$ and go to (b).

Proposition 2: If x^1 has property (c), then the ideal point x_1^{*1} in Ω_{x^1} has the property that there exists $i \in I_{x^\circ}$, $i \notin I_{x^1}$, and $I_{x^1} \subseteq I_{x^\circ}$.

Proof: Note that if $i_0 \in I_{x^1}$, then $x^1_{i_\circ} < x^{*1}_{i_\circ}$.

Note also that $x^1 \geqslant x^\circ$, therefore $\Omega_{x^1} \leqslant \Omega$, i.e.

$$x^{*1}_{i_\circ} = \sup_{x \in \Omega_{x^1}} pr_{i_\circ}(x) \leqslant \sup_{x \in \Omega_{x^\circ}} pr_{i_\circ}(x) = x^*_{i_\circ}.$$

It follows that $x_{i_\circ} \leqslant x^1_{i_\circ} < x^{*1}_{i_\circ} \leqslant x^*_{i_\circ}$, i.e. $i_\circ \in I_{x^\circ}$.
Therefore $I_{x^1} \leqslant I_{x^\circ}$.

Now, let us prove that there exists $i \in I_{x^\circ}$ with $i \notin I_{x^1}$. Bearing in mind the determination of the ideal point

$\forall i \in \{1, 2, \ldots, n\}\ \exists y^i \in \Omega$ such that $y^i_i = x^{*1}_i$ and $y^i \geqslant x^1$.

Assume that for each $i \in I_{x^\circ}$, $y^i_i = x^{*1}_i > x^1_i$, a that $i \in I_{x^1}$. Then, taking

$(\lambda_i)_{i \in I_{x^\circ}}$ such that $\sum_{i \in I_{x^\circ}} \lambda_i = 1$ it follows $y = \sum_{i \in I_{x^\circ}} \lambda_i y^i \in \Omega$, Ω convex.

$$y_j = \sum_{i \in I_{x^\circ}} \lambda_i y^i_j \qquad j = 1, \ldots, n$$

$$y^i_j \geqslant x^1_j \qquad \forall j = 1, \ldots, n \qquad \forall i \in I_{x^\circ}, \quad i \neq j$$

$$y^j_j > x^1_j \qquad j \in I_{x^\circ}$$

i.e. $\forall j \in I_{x^0}$, we would have $y_j > x_j^1$ and $y = (y_1, ..., y_n) \in \Omega_{x'}$, therefore $d_{x^*}^{x^0}(y) > d_{x^*}^{x^0}(x^1)$ which contradict the optimality of $x^1(e)$.

Corollary: The procedure has a finite number of steps.

Proof: According to proposition 2, $|I_x|$ decreases. After a finite number of steps for a certain $x \in \Omega$, we have $I_x = \emptyset$. Therefore the obtained point is maximal.

Remark: The above algorithm can be applied to the determination of a point for the function $f : \mathbf{R}^n \to \mathbf{R}^p$ on the domain $\Omega \neq \emptyset$, $\Omega \leqslant \mathbf{R}^n$, assuming that $f(\Omega) \leqslant \mathbf{R}^p$ is convex and compact. In this framework is the following situation:

1. $\Omega \neq \emptyset$, $\Omega \leqslant \mathbf{R}^n$ convex and compact
2. $f : \mathbf{R}^n \to \mathbf{R}^p$ $f = (f_1, f_2, ..., f_p)$,
   $f_i : \mathbf{R}^n \to \mathbf{R}$ $i = 1, ..., p$, linear functionals.

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METHODE DE SELECTION D'UN POINT D'OPTIMUM SELON PLUSIEURS CRITERES

Les auteurs présentent un procédé de détermination d'un élément maximal pour certains sous-ensembles de $\mathbf{R}^n$. On cherche un élément maximal qui augmente un point donné; il existe une procédure de travail et on cherche la modalité de l'améliorer. L'idée de sélection est basée sur la distance par rapport au point idéal, choisi pour correspondre aux notions fuzzy. Le choix fait permet de trouver, dans le cas convexe, un algorithme à nombre fini de pas menant à un élément maximal.

МЕТОД ОТБОРА ОПТИМАЛЬНОЙ ТОЧКИ ПО НЕСКОЛЬКИМ КРИТЕРИЯМ

В работе приведён способ определения максимального элемента для некоторых подмножеств из $\mathbf{R}^n$. Идёт поиск максимального элемента, увеличивающего данную точку; есть процедура, которой мы пользуемся, и идёт поиск её улучшения. Идея отбора основывается на расстоянии, относительном идеальной точки, выбранной таким образом, чтобы соответствовала понятиям фаззи. Сделанный выбор позволил найдти, в выпуклом случае, алгоритм с конечным числом шагов, приводящих к максимальному элементу.

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