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Economic System Structure Hierarchization

The triangulation problem occured two decades ago in the empirical studies. Since that time a series of empirical algorithms have appeared, meant to solve this problem ([2], [4], [5]). In the present paper we shall prove a theorem that limits to the maximum the field wherein the problem solutions must be sought for. By this theorem we shall describe an accurate algorithm for solving the triangulation problem.

1. INTRODUCTION

The description of economic systems, in most cases, is achieved under the form of a linear model. In the framework of such a system we shall discern "n" interconnected elementary systems.

The element $e \in I$ (subsystem, block, activity branch, operating process, productive unit a.s.o.), $I = \{1, 2, ..., n\}$ being an ordered family of indices associated to the elementary systems, is called an elementary system of system S. The connexions between the elementary systems appear like flows. These flows can have various concrete forms, as: product flows (services, commercial transactions), money flows, information flows, production relations.

In this study we call *structure* of a production economic system the set of the flows between the elementary systems.

Let X be the structure matrix of complex economic system with $n \times n$ size:

$$X = (x_{ij})_{i,j \in I}; \qquad x_{ij} \geqslant 0. \tag{1}$$

The structure of a sufficiently evolved and complex economic system is non-decomposable.

At the same time theoretical and empirical investigations prove that, as a rule, the system's structure is hierarchically ordered. Thus, the structure matrix X may be ordered such that the Helmstädter criterion be satisfied:

$$\sum_i \sum_{j<i} x_{ij} \longrightarrow max \tag{2}$$

or

$$\sum_i \sum_{j>i} x_{ij} \longrightarrow max \tag{3}$$

that is to say, the sum of the flows under the main diagonal (2) or that of the flows over the diagonal (3) must be a maximum.

When the hierarchized structure is not obvious through the real existence of the system, or when it is insufficiently known, a description which could satisfy criteria (2) or (3) cannot be achieved.

The reordering operation of the structure description is known under the name of *triangulation.* A complex analysis of the triangulation algorithms is due to B. Korte and W. Oberhofer [4]. A procedural method is presented by M. Capătă [2].

In a non-decomposable structure matrix X, the symmetric elements (x_{ij}, x_{ji}) form a couple. Through hierarchization, the couple components remain inseparable, and only their order in its framework is changing (as against the symmetry axis $i = j$). In a decomposable system the flows appear as isolated elements, non-organized in couples.

The concept of structure ordering implies a directioning of the flows. We agree to call *hierarchical ordering*, that ordering which satisfies criterion (2) or the criterion equivalent to the latter

$$\sum_i \sum_{j>i} x_{ij} \longrightarrow min. \tag{4}$$

2. A THEORETICAL SUBSTANTIATION OF THE TRIANGULATION PROBLEM

Let us consider an antisymmetrical graph $G = (V, A)$, together with a value-function of the arcs:

$$f : A \rightarrow R_+.$$

To each total order T of set V of the graph vertexes an "evaluation" is naturally associated:

$$\varepsilon(T) = \sum_{\substack{u>v \\ T}} f[u, v]. \tag{5}$$

The triangulation problem consists in determining those total orders T^* which minimize function ε, i.e.:

$$\varepsilon(T^*) = \min_T \varepsilon(T).$$

Let us now consider that vertex set V of the graph G is finite and, for simplifying, let $V = \{1, 2, ..., n\}$.

Any total order T of V is uniquely determined by a permutation p of set V, through $p(1) \underset{T}{<} p(2) \underset{T}{<} \ldots \underset{T}{<} p(n)$ and hence through an element of the symmetrical group $Sim(n)$. Thus, the triangulation problem consists in determining the permutations p^* called optimals which satisfy:

$$\varepsilon(p^*) = \min_{p \in Sim(n)} \varepsilon(p). \tag{6}$$

Let $B = (b_{i,j})$ denote the boolean (antisymmetrical) matrix of graph G. By extending, if needed, with 0 the value-function f, we may obviously write that:

$$\varepsilon(p) = \sum_{1 \leqslant j < i \leqslant n} b_{p(i),\, p(j)}\, f[p(i), p(j)].$$

Let now **B** be Boolean algebra $\{0, 1\}$. Considering $n(n-1)/2$ generators $(g_{i,j})_{1 \leqslant j < i \leqslant n}$, let us denote by C the free Boolean algebra, generated by them.

There evidently is an application $e : C \to R_+$ that extends the associations $g_{i,j} \to f[i, j]$ and also, to each permutation $p \in Sim(n)$ an element from C can be associated, as follows:

$$\varphi(p) = \sum_{1 \leqslant j < i \leqslant n} b_{p(i),p(j)} g_{p(i),p(j)}.$$

Definition 1. We shall say that a permutation p is *atomic* if $\varphi(p)$ is an atom in the lattice $L = Im(\varphi)$.

Corollary. The optimal permutations are atomic.

It is evident, because application e is isotonic.

Definition 2. We shall say that a permutation p is *hamiltonian* if for any $i = 1, 2, \ldots, n-1$ we have $b_{p(i),p(i+1)} = 1$ (in other words, permutation p determines a hamiltonian path in graph G).

Theorem 1. A permutation is atomic if and only if it is hamiltonian.

Proof. Let p be an atomic permutation and let us suppose it is not hamiltonian. Then there is an i minimal, with the property that $1 \leqslant i \leqslant n-1$ and $b_{p(i),p(i+1)} = 0$. As graph G is antisymmetrical, it results that $b_{p(i+1),p(i)} = 1$. On considering permutation q obtained from p through multiplication with transposition $(i, i+1)$, we notice that:

$$\varphi(p) = \varphi(q) + g_{i,i+1} > \varphi(q)$$

and hence the hypothesis is contradicted.

Reciprocally, let us suppose that p is a hamiltonian permutation. By hypothesis $b_{p(i),p(i+1)} = 1$ for $i = 1, 2, \ldots, n-1$, hence in $\varphi(p)$ the components $g_{p'i),p(i+1)}$ do not appear (we shall consider everywhere that $g_{i,j} = g_{j,i}$).

Let q be another permutation such that $\varphi(q) \leqslant \varphi(p)$ in lattice L. Let $p(1) = q(k_1)$, $p(2) = q(k_2)$, ..., $p(n) = q(k_n)$.

If $k_1 > k_2$, then $b_{q(k_1),q(k_2)} = b_{p(1),p(2)} = 1$, hence in $\varphi(q)$ the component $g_{p(1),p(2)}$ appears, — contradiction. Hence $k_1 < k_2$.

If $k_2 > k_3$, then $b_{q(k_2),q(k_3)} = b_{p(2),p(3)} = 1$, hence in $\varphi(q)$ the component $g_{p(2),p(3)}$ appears, contradiction. Hence $k_2 < k_3$ and analogically $k_i < k_{i+1}$ for $i = 1, 2, ..., n-1$.

But $1 \leqslant k_i \leqslant n$ for an $i = 1, 2, ..., n$. $k_i = i$ for $i = 1, 2, ..., n$ results as unique possibility and hence $p = q$. Consequently p is atomic and the theorem is proved.

3. ON THE STRUCTURE OF NON-DECOMPOSABLE SYSTEMS

The experience of triangulation proved that there are interdependences and reciprocal flows between branches. Hence, a perfect triangulation, the transformation of all connections into linear dependences, cannot be achieved in the statistical balances. That means that, through the very intrinsic nature of the economic system, the matrix of flows (x_{ij}) and the matrix of the input coefficients A are non-decomposable, i.e. they cannot be brought under the triangular or quasitriangular form. Consequently, the triangulation criterion must be worded so as to take into account this property of the matrix. Moreover, one must take into account that from the empirical tables a certain hierarchical order can arise, that reveals a polarization of the connections, a concentration of the interbranch flows.

In the non-decomposable system S, there are couples wherein both components are significant. Between these components one of the following relations holds:

$$x_{ij} = x_{ji} \tag{7}$$

$$x_{ij} < x_{ji}, \tag{8}$$

$$x_{ij} > x_{ji}, \tag{9}$$

The components of relation (7) are not affected by the hierarchical ordering of the structure. If the former are to be found in relation (8), the permutation $x_{ij} \leftrightarrows x_{ji}$ must be carried out so as to achieve criterion (2), and if the components are there in relation (9), thus criterion is satisfied.

If during the re-ordering process the desintegration of the couple (equivalent to the violation of the real structure) could be possible, we could bring the larger component of the couple under the diagonal, and the smaller component over the diagonal. In this case, all couples would be in relation (9) and consequently the system would be perfectly ordered, achieving in this way the maximum possible value of sum:

$$S^+ = \sum_i \sum_{j<i} max(x_{ij}, x_{ji}). \tag{10}$$

When ordering the description of the system, such a hypothesis cannot be adopted. The re-ordering must be carried out through symmetrical permutation that preserves the present system structure. Besides relations (7) and (9) there appear also type (8) relations, that introduce so called *reversals* and *contradictions* within the system's structure.

For the differences matrix X_D we calculate

$$D = \sum_{i \in I} \sum_{\substack{j \in I \\ j<i}} |\alpha_{ij}| = \sum_{i \in I} \sum_{\substack{j \in I \\ j<i}} |x_{ij} - x_{ji}| \tag{11}$$

and in [8] the following.

Theorem is given: irrespective of the elementary order, we have from system S:

$$D = \sum_{i=j+1}^{n} \sum_{j=1}^{n-1} max(x_{ij}, x_{ji}) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} min(x_{ij}, x_{ji}).$$

To state precisely the notion of system-S component ordering, we define the elementary system *rank* right from the start. In figure 1 a system is presented which is ordered according to criterion (2), where:

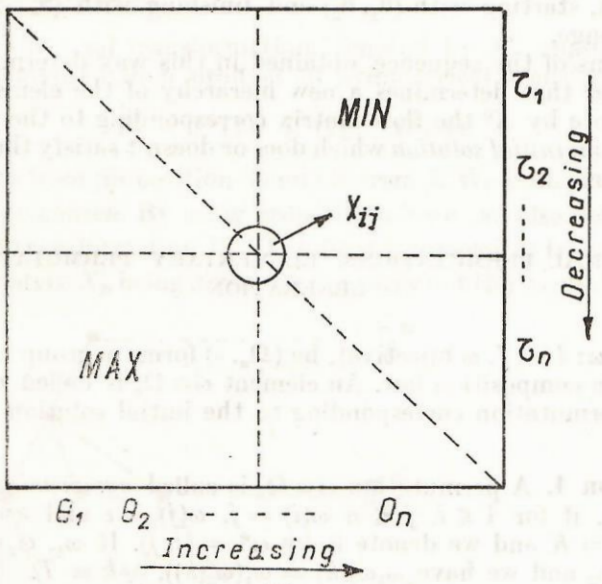


Fig. 1

$$\tau_i = \sum_{j=1}^{n} x_{ij};\ i \in I,\ x_{ii} = 0 \tag{12}$$

$$\theta_j = \sum_{i=1}^{n} x_{ij};\ j \in I,\ x_{ii} = 0 \tag{13}$$

and the sequences $\{\theta_j\}_{j \in I}$, $\{\tau_i\}_{i \in I}$ are increasingly and decreasingly, respectively, ordered.

Only in an ideal case the order of both sequences obtained through (12) and (13) is identical (that is, a term or an elementary system $i \in I$ occupies

the same rank in the two sequences). Let us compare the ranks of the elements of sequences $\{\theta_i\}_{i\in I}$, $\{\tau_j\}_{j\in I}$. We consider the application:

$$\varphi : T_0 \longrightarrow T_0'$$

where T_0 represents the set of the ranks corresponding to the increasingly ordered sequence

T_0' the set of the ranks corresponding to the decreasingly ordered sequence.

Definition 3. We shall say that between the ranks of the two sequences a "contradiction" appears, if there is only one $l\in I$ such that $\varphi(l) \neq l$. We have rank (θ_i) = rank (τ_j) if $\varphi(k) = k$, irrespective of $k\in I$.

The sequence $\{\theta_i\}_{i\in I}$ will be ordered increasingly, by analysing the elements adjacent as rank. This will be changed if they do not correspond to the desired order, letting them on their places if they correspond. The ordering is ended when, starting with (θ_1, θ_2) and finishing with (θ_{n-1}, θ_n), we don't make any change.

The terms of the sequence obtained in this way determine a new rank order, an order that determines a new hierarchy of the elementary systems $e\in I$. We denote by X^0 the flow matrix corresponding to the hierarchy. We thus obtained an *initial solution* which does or doesn't satisfy the initially enuntiated solution.

4. SYMMETRICAL PERMUTATIONS, ELEMENTARY PERMUTATIONS. THEIR GENERATION

Let $\Omega_n = \{\omega/\omega: I \to I, \omega \text{ bijective}\}$, be $(\Omega_n, \circ)$ forms a group with respect to "$\circ$" application composition law. An element $\omega\in\Omega_n$ is called *n-degree permutation.* The permutation corresponding to the initial solution X^0 is identical permutation.

Definition 4. A permutation $\omega\in\Omega_n$ is called *symmetrical* (symmetrical transposition), if for $1 \leqslant i, j \leqslant n$ $\omega(i) = j$, $\omega(j) = i$ and $\forall K \in I$, $K \neq i$, $K \neq j$, $\omega(K) = K$ and we denote it by $\omega^* = (i, j)$. If $\omega_1, \omega_2\in\Omega_n$, we write $\omega_1\circ\omega_2 = \omega_1 \cdot \omega_2$ and we have $\omega_1\omega_2(k) = \omega_1(\omega_2(k))$, $\forall k \in T$.

Proposition 1 [8]. If $\omega_1(k)\in T_0$, $\omega_2(k) \in T_0$, $k \in T$, then relation $\omega_1(\omega_2(k)) = \omega_2(k)$ is equivalent to definition 5 (rank (θ_i) = rank (τ_j)).

Proposition 2. The change of two lines (columns) in the matrix X^0 is equivalent to the multiplication at *LH* (*RH*) side of the matrix, with the unit matrix wherein the same lines (columns) were changed between them.

The proof is easily made, by using the direct algebraic calculation. Let we denote this matrix by T^* (symmetrical transformation matrix).

Proposition 3 [8]. The symmetrical permutation ω^* is equivalent to a symmetrical transformation T^*.

It results from definition 4 that a symmetrical permutation modifies not only the position of a couple, but also the position of the couples corres-

ponding to two elementary systems [8]. The re-ordering of structure S is to be preferably done through elementary permutations.

Definition 5. The symmetrical permutation of two elementary systems is called elementary permutation. Let $i\omega_t, j\omega_t$ be two elementary systems which change their place in the ω_t (t being the iteration).

Theorem 2. If $i\omega_1 \neq \ldots i\omega_s$, $j\omega_1 \neq \ldots j\omega_s$, then the elementary permutations $\omega_1^*, \omega_2^* \ldots \omega_s^*$ are included within a permutation $\Pi_{t'}^*$ ($t' = 1, 2, \ldots, N'$, $N' \subset N$), which, applied to the matrix X_D°, leads to the same result, obtained by successively applying the elementary permutations $\omega_1^*, \omega_2^* \ldots \omega_s^*$ to the matrix X.

The proof of this theorem results from:

Theorem 3. Any permutation $\omega \in \Omega_n$ is decomposed in a product of symmetrical permutations [3].

Remark. The final transformation, denoted by T_f^*, applied to the matrix X_D° is decomposed in a product of symmetrical transformations, i.e.:

$$T_f^* = T^*(1).\ T^*(2) \ldots T^*(N).$$

The proof results from proposition 3 and theorem 3. We shall call permutation $\Pi_{t'}^*$ a *transfer permutation*. By using proposition 3 we can also define the transit symmetrical transformation $\Pi_{t'}^*$. The final symmetrical transformation T_f^* applied to the matrix X_D° being given, we can carry out the permutation decom-

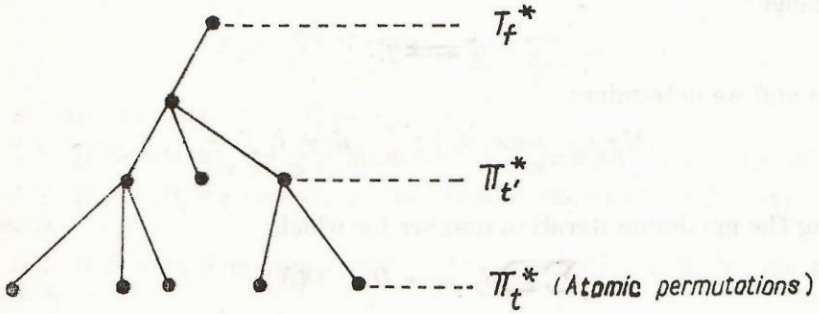


Fig. 2

position as in Fig. 2, and where from the transit possibilities presented in Fig. 3 result.

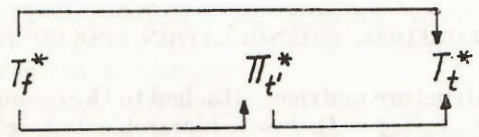


Fig. 3

When working with Π_t^* $(t = 1, 2, ..., N')$ transformation, the convergence is faster, but the computer programming gets complicated. We are fo a procedure using elementary permutations.

The conditions imposed by the elementary permutation procedure [8 lead to a "gradient"-type method, used in the structural hierarchization o system S, according to that presented in Fig. 4.

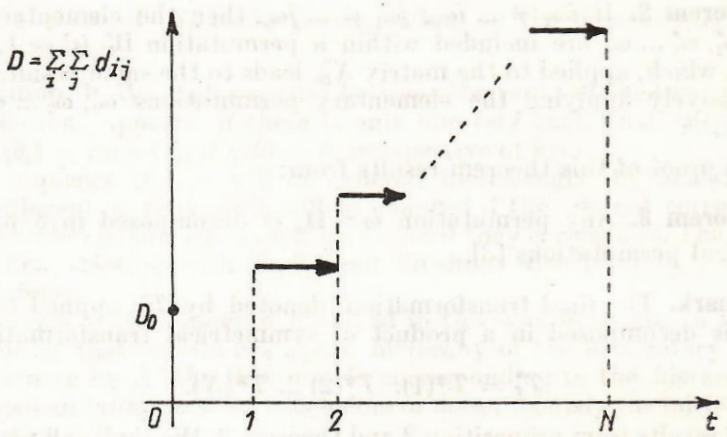


Fig. 4

Further on, the question is: how can one determine the ranks i_t^*, j_t^* which generate the symmetrical transformation and lead, in iteration "t", to the change:

$$i_t^* \rightleftarrows j_t^*.$$

To this end we determine:

$$M_{i_t^* j_t^*} = \max_{\substack{dij<0 \\ j<i}} |\, d_{ij} \,| \; ; \qquad i_t^* \neq i,\ j_t^* \neq j \quad (1 \leqslant t \leqslant N)$$

N being the maximum iteration number for which:

$$\sum_i \sum_j d_{ji} \longrightarrow D = D(N).$$

Note. The sequence $\mathfrak{M}_t = \{M_{i_t^* j_t^*}\}_{1 \leqslant t \leqslant N}$ is increasing and upper bounded by "0". We have:

$$\mathfrak{M}_t \xrightarrow{t \to N} 0.$$

5. A QUASI-OPTIMAL TRIANGULATION PROCEDURE

As a rule, the structure matrices attached to the economic systems satisfy the property: $X_{ij} \neq X_{ji} (i, j \in I)$. For a hierarchizated ordering of the structure of such systems we suggest the following procedure:

1. We agree that the larger element from the couple $max(X_{ij}, X_{ji})$ be marked by an asterisk.

Note. If both elements of the couple are equal, neither will be marked.

2. We determine how many marked elements there are under the diagonal (n_s^*) and above it (n_d^*), where:

$$0 \leqslant n_s^* + n_d^* \leqslant \frac{n(n-1)}{2}.$$

3.1. If $n_s^* < n_d^*$, the upper triangulation will be carried out, that is to say one will try that all, or as many as possible, marked elements under the diagonal be transferred above it.

3.2. If $n_s^* > n_d^*$, the lower triangulation will be carried out, that is to say we will try that all, or as many as possible, marked elements above the diagonal be transferred under it.

3.3. If $n_s^* = n_d^*$, we arbitrarily select the triangulation manner (we agree, for economical reasons, to select the lower triangulation).

4. We read from north-west towards south-east the elements

$$X_{ij}(j < i).$$

For the position (i^*, j^*) of such an element, we determine:

$$S_1 = \sum_{i=i^*}^{i^*} \sum_{j=i^*}^{j^*} X_{ij} + \sum_{i=i^*+1}^{j^*} \sum_{j=j^*}^{j^*} X_{ji}$$

$$S_2 = \sum_{i=i^*}^{j^*} \sum_{j=i^*}^{i^*} X_{ij} + \sum_{i=j^*}^{j^*} \sum_{j=i^*+1}^{j^*} X_{ji}.$$

5. We calculate: $S = S_1 - S_2$.

5.1. If $S > 0$, we make a permutation $i^* \rightleftarrows j^*$. We go over to step 4.

5.2. If $S < 0$, we go over to the next marked element X_{ij}. We go over to step 5.

5.3. If $S = 0$, it is immaterial if the permutation is done. We go over to step 4.

6. When working with finite quantities and as the improving criterion determined by sum S exists, the calculation process is stopped if:

$$|S^+ - S^{(t)}| < \delta,$$

where δ is an improved value ($\delta \cong 10^{-4}$), t the procedure iteration.

When, following an analysis of all elements, marked by asterisk above the diagonal, we can no longer effect any permutation, the algorithm is stopped, and a suboptimum is obtained.

Starting from logical considerations we can point out that we can arrive to the same order, if we agree to mark the smallest element from the couple $(min(x_{ij}, x_{ji}))$.

Systems can be identified that have the structure matrices formed from elements which satisfy the property:

$$x_{ij} \approx x_{ji}$$

i.e. the couple elements differ very little. So, as to achieve the ordering in this case, we resort to the matrix of the differences: $X_D = (d_{ij})$, $i, j \in I$. These differences are analysed in terms of a threshold ($\bar{p}$), [15] imposing $d_{ij} \leqslant \bar{p}$, $i, j \in I$. We can prove the equivalence between the procedure with marked elements and the procedure with differences, whose mechanism was explined in [8].

6. COMBINATORIAL ALGORITHM FOR CARRYING OUT THE TRIANGULATION PROBLEM

From the theorem 1 it results that, through the enumeration of the hamiltonian permutation we implicitly obtain an enumeration of the optimal permutations. The hamiltonian permutation enumeration can be made with an accurate algorithm of the "branch-and-bound" type [4], whose description will be given further down.

Ist phase. With a view to diminishing the iterative steps necessary to solve the problem, we proceed to an initial ordering of set V of graph G vertexes.

Step 1. If graph G has no vertexes, the problem is trivial; otherwise we set $k = 1$, $m = n$, $H = G$.

Step 2. If graph H has no vertexes, that means that the initial order determines the unique optimal permutation, stop. Otherwise we go further with:

Step 3. By exploring the boolean matrix associated to graph H, we determine the inner halfdegree $i(v)$ for each vertex v of graph H.

Step 4. If the inner halfdegree $i(v)$ of a vertex v is equal to m, we assign to v the place m in the initial ordering, then we put $n = m - 1$ and go on with step. 6. Otherwise:

Step 5. If the inner halfdegree $i(v)$ of a vertex v is equal to 0, we assign to v the place k in the initial order, put $k = k + 1$ and go on with step 6. Otherwise we assign to the vertexes of H order numbers between k and m, according to the increasing values of their inner halfdegrees; we go on with the IInd phase.

Step 6. We exclude vertex v and we denote the graph left after the exclusion by H. We go over to step 2.

IInd phase. The vertexes of G, eliminated during phase I, are no longer taken into account, their places in the initial ordering remaining unchanged in any optimal ordering (these vertexes will be called degenerated, and graph G will be called degenerated, in case such vertexes appear).

Step 7. We put $\varepsilon^* = \infty$, $s = k$, $i = 0$.

Step 8. We put $t = s$ and $J(r) = 0$, for $k \leqslant r \leqslant m$.

Step 9. We put $i = i + 1$, $p(i) = t$ and we mark column t of the boolean matrix associated to the graph.

Step 10. If $i < m$, we go on with step 12. Otherwise p is a hamiltonian, hence atomic permutation, and $\varepsilon(p)$ is estimated.

Step 11. If $\varepsilon(p) < \varepsilon^*$, we put $\varepsilon^* = \varepsilon(p)$ and retain permutation p, then go on with step 13. If $\varepsilon(p) \geqslant \varepsilon^*$, we go on with step 13.

Step 12. We explore line $p(i)$ in the matrix of graph H, beginning with column $j(i) + 1$. In the first non-marked column t with $b_{p(i),t} = 1$ we put $j(i) = t$ and go further from step 9. If such columns do not exist, then we go on with:

Step 13. We annul the marking of column $p(i)$; we put $i = i - 1$.

Step 14. If $i > 0$, we go on with step 12, otherwise $s = s + 1$ and we go on with:

Step 15. If $s \leqslant m$, we go on with step 8. Otherwise, stop.

Finally, we have retained in ε^* the value $\varepsilon(p^*)$ and moreover one of the optimal permutations was also retained. The algorithm can be easily slightly modified, so as to retain all optimal permutations.

This algorithm is easily programmable in the FORTRAN or PL/1 language and is independent of the type of graph G. For instance, for $n = 4$, there are four types of graphs [1], determined by the anti-symmetrical Boolean matrices:

$$\begin{pmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&1\\ 0&0&1&1\\ 1&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&1\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&0\\ 0&0&1&1\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix}$$

among those only the last type is non-degenerated. The chances that a graph G be one of these types are 3/8, 1/8, 1/8, 3/8, respectively, calculated by hypothesis that all values $f(a)$ are uniformly distributed in the interval [0, 1]. Also, the non-degenerated type IV allows a number of 5 atomic permutations, viz.:

$$1, \quad (1234), (1432), (1243), (13)\,(24).$$

whose chances of being optimal permutations are (calculated in the same hypothesis on the arc values): 491/720, 81/720, 81/720, 66/720, 1/720.

Unfortunately at the same time with the increase of number n of graph vertexes, the non-degenerated type number increases approximately with speed $2^{\frac{n-1}{2}}$, and the atomic permutation maximal number increases approximately with speed 2^{n-2}, that renders the algorithm non-adequate for big

numbers n. As each atomic permutation is likely to be optimal (for certai graphs) it seems that there do not exist practical possibilities to solve exactl the triangulation problem for a big n.

The tests carried out by various authors with graphs resulted from ba lances of the connections between branches of some state economies [5], [8 seem to point out the fact that these graphs have a special structure.

Finally, we can determine various structure indicators whose formula tion and synthesis is presented in [6], [7].

The hierarchized-structure economic systems (triangulated, diagonalized structure matrices etc.) have various applications in macroeconomic structura analyses, numerical approximations [7], informational system rationalizatio prognosis and planning calculations, economic-growth cycles directing [6] The utilization of such systems in the informational (entropic) analysis o complex and of hierarchical systems [8] is not deprived of interest.

The most traditional applications are met within the *input-output ana lysis*,— territorial or national level.

Let us assume an economic system whose structure matrix (matrix o differences) is:

$$X_D^0 = \begin{bmatrix} 0 & 5 & 2 & -3 & 1 \\ 0 & 0 & 6 & 4 & 8 \\ 0 & 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We apply to this matrix the transformation:

$$T_1^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which carries out the change: I $\rightleftarrows$ IV. By effecting

we obtain: $$X_1 = T_1^* . X_D^0 \cdot T_1^{*\prime}$$

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 10 \\ 4 & 0 & 6 & 0 & 8 \\ 7 & 0 & 0 & 0 & 9 \\ -3 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the quasi-optimal procedure we want to carry out change: I$\rightleftarrows$III

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 10 \\ 4 & 0 & 6 & 0 & 8 \\ 7 & 0 & 0 & 0 & 9 \\ -3 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The framed areas in the matrix will be affected by the permutation, and the efficiency of the latter is determined by the following calculation:

$$S_1 = \sum_{i=1}^{1}\sum_{j=1}^{3} d_{ji} + \sum_{j=2}^{3}\sum_{j=3}^{3} d_{ji}$$

$$S_2 = \sum_{i=1}^{3}\sum_{j=1}^{1} d_{ij} + \sum_{j=3}^{3}\sum_{j=2}^{3} d_{ij}$$

$S = S_1 - S_2 < 0$, and this means that the permutation is to be done. This is the working mode of the presented quasi-optimal algorithm. The number of generated permutations being big enough, we did not consider a concrete problem to be carried out. When using a medium-capacity computer, the problem is acceptable up to a matrix dimension of 24×24.

We shall present further the triangulation mode for an economic system with 8 elementary systems (branches). The initial matrix is given in Table 1.

Table 1

Consuming branches / Producing branches	1	2	3	4	5	6	7	8
1	0	0.24	0	0	0.015	0	0.29	0
2	0	0.62	0	0.73	0	0.15	0	0
3	0.82	0	0	0	0	0.65	0	0.68
4	0	0.55	0	0.36	0	0	0	0
5	0	0	0.41	0	0	0	0.13	0.22
6	0	0.15	0	0	0	0	0	0
7	0	0	0	0	0.18	0	0	0
8	0	0	0.59	0.71	0	0	0	0.83

The Boolean matrix attached to this technological coefficients table is the following one:

$$A = \begin{bmatrix} 0\,1\,0\,0\,1\,0\,1\,0 \\ 0\,1\,0\,1\,0\,1\,0\,0 \\ 1\,0\,0\,0\,0\,1\,0\,1 \\ 0\,1\,0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,1\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0\,1 \end{bmatrix}$$

By using the presented algorithm, the found final permutation is

$$p_f^* = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,5\,7\,3\,8\,6\,2\,4 \end{pmatrix}$$

This is a "compressed" permutation as it is the result of the product of some elementary symmetrical permutations. By applying to matrix A the determined permutation, we obtain the triangulated matrix:

$$A_T = \begin{bmatrix} 0\,1\,1\,0\,0\,0\,1\,0 \\ 0\,0\,1\,1\,1\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,1\,1\,0\,0 \\ 0\,0\,0\,1\,1\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1\,1\,1 \\ 0\,0\,0\,0\,0\,0\,1\,1 \end{bmatrix}$$

Notes. a) If graph has loops (corresponding to the non-zero elements on the main diagonal (internal consumption)), these will be kept for any permutation.

b) Concerning the presented example, the obtained permutation is optimal; any other permutation will have at least 5 elements under the diagonal, as the graph has 5 independent circuits, of a length greater than 1, and all the other circuits are obtained from those. For the presented algorithm, a FORTRAN programme was drawn up, for a matrix having the dimension 45 × 45. The algorithm detects also graph cycles, giving also the

corresponding message, and it allows the triangulation in two manners (above and under the main diagonal). The matrix is also edited under its disorganized form and then the matrix under triangulated form is edited. So as to symplify, to each line (column) a numerical code of 2 digits was assigned, for identifying the correspondence:

$$a^0_{ij} \longrightarrow a^T_{ij}$$

through the agency of the Boolean matrix.

The triangulated matrices are also used for *rationalizing informational systems*, in whose framework the documents passing through various points of advising, processing, sorting out etc. must not form cycles. If we denote by D the set of documents and by S the set of departments wherein they move on, we can achieve Card (S) = Card (S) = n through addition in the line (column) matrix with null elements. The initial matrix will have the 6×7 size. Through addition of a line (fictious document) we obtain, as initial matrix, (with 7×7 size) corresponding to Table 2.

Table 2

Departments / Documents	S_1	S_2	S_3	S_4	S_5	S_6	S_7
D_1	0	0	1	1	0	0	1
D_2	0	0	0	0	0	0	0
D_3	0	0	0	0	1	1	0
D_4	0	1	0	0	1	0	0
D_5	0	0	0	0	0	0	1
D_6	0	1	0	0	0	0	1
D_7	0	0	0	0	0	0	0

The existence of an element $b_{ij} = 1$ means an action on document "i" in department "j".

The found optimal permutation is:

$$p^*_f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 5 & 6 & 3 & 4 & 1 \end{pmatrix}.$$

This one, applied to matrix B^0 leads to the obtaining of matrix B_T:

$$p^*_f \cdot B^0 = B_T$$

where

$$B_T = \begin{bmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 1 & 0 & 0 & & & & \\ 1 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The triangulated matrices find applications in working out certain input-output models for planning (programming) the production, in balancing models of production capacities, in solving some technical problems in the field of *telecommunications*, etc.

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HIERARCHISATION DE LA STRUCTURE DES SYSTEMES ECONOMIQUES

Le problème de la triangulation a été posé voilà vingt ans dans les études empiriques. Depuis, toute une série d'algorithmes empiriques ont été conçus pour la résolution de ce problème ([2], [3], [4]).

Dans le présent ouvrage les auteurs donnent la démonstration d'un théorème qui limite au maximum le domaine où l'on doit chercher les solutions du problème et, sur la base de ce théorème, ils décrivent un algorithm exacte pour résoudre le problème de la triangulation.

ИЕРАРХИЧЕСКОЕ УПОРЯДОЧЕНИЕ СТРУКТУРЫ ЭКОНОМИЧЕСКИХ СИСТЕМ

Проблема триангуляции была сформулирована два десятилетия назад в эмпирических исследованиях типа вход—выход для различных стран. С тех пор появился ряд эмпирических алгоритмов для разрешения этой задачи ([2], [3], [4]). В настоящей работе мы продемонстрируем теорему, максимально лимитирующую область в которой нужно искать решения задачи и, на базе этой теоремы, опишем точный алгоритм решения задачи триангуляции.

30. VI. 1976.