DAN VĂIDEANU
Bucharest University

On a Markovian Inventory Model

The author presents in the paper "On a Markovian Inventory Model" a general random flowchart modelling the situations analysed in the chapter devoted to inventory theory from operations research. Author's variant shows the operational results which from the point of view of applications are of great interest.

1. THE MODEL

We call "stock system" a set of objects or people, with random withdrawals, which must be reorganized at certain moments, depending on these demands.

Solving separately each inventory problem is more and more difficult, but many different situations in appearance, are similar when they are formulated in relationships.

So, mathematical approaches come to complete the experience and intuition of a stock manager.

They are many aims we can reach using this approach.

First, we might be interested to the management of a stock system (that means satisfying demands and carrying items in our inventory) looking for the optimal reorder quantity (input) which must satisfy a deterministic or random demand. In the most cases, to compute that optimal value means to minimize a cost function or to maximize a profit one.

Second, we may ignore computing any optimal value, but be interested to the operational characteristics of that system, which may be obtained from the equations of mathematical model and can not be determined in other ways.

Choosing this second aim, we will try in the following sections to establish some operational characteristics of a given stock system by means of Markov chains theory.

Let us now consider the case of a stock system [1], where the inventory level is given by the next relationship:

$$I_n = I_{n-1} + Q_n - R_n, \qquad n = 1, 2, \ldots \tag{1.1}$$

The following symbols are used:

— I_n — inventory level at the end of the present period, n;

— Q_n — reorder quantity (input) during the same period;

— R_n — demand (output or withdrawal) satisfied during this period.

In (1-1) the demand R_n is the term which influences the variation of the others. It may be a deterministic variable or a random one and this is the most interesting situation.

We may know in this case the class of distribution or even its parameters.

When we have no information about this random demand, the distribution and parameters can be estimated by statistical methods [2].

For the following discussions it is necessary to assume that:

$$Q_n = Q_n\ (I_n, R_n).$$

Suppose that R_n is a discrete random variable with independent values and known distributions given by:

$$P\{R_n = k\} = p_k. \tag{1.2}$$

Let us now consider the sequence of random variables $\{I_n\}_{n \in N}$ and a set S, of all the values that $\{I_n\}_{n \in N}$ can get.

One may see that this sequence of random variables satisfies the Markov Property, that is:

$$P(I_n = i_n \mid I_{n-1} = i_{n-1}, \ldots, \quad I_0 = i_0) = P(I_n = i_n \mid I_{n-1} = i_{n-1}) \tag{1.3}$$

in other words, the whole history reduces at the previous step.

In fact, using (1.1) we can write:

$$\begin{gathered} \{I_0 = i_0\} \supset \{I_1 = i_1\} \supset \ldots \supset \{I_{n-1} = i_{n-1}\} \supset \ldots \\ n \in N, \qquad i_0, i_1, \ldots, i_{n-1} \ldots \in S \end{gathered} \tag{1.4}$$

and then:

$$\begin{gathered} P(I_n = i_n \mid I_{n-1} = i_{n-1}, \ldots, I_0 = i_0) = \\ = P(I_n = i_n \mid \{I_{n-1} = i_{n-1}\} \cap \ldots \cap \{I_0 = i_0\}) = \\ = P(I_n = i_n \mid I_{n-1} = i_{n-1}) \end{gathered} \tag{1.5}$$

so the sequence $\{I_n\}_{n \in N}$ of random variables has the Markov Property.

Moreover, let us note that the sequence $\{I_n\}_{n \in N}$ is a chain with independent increments.

The set S of the possible values of $\{I_n\}_{n \in N}$ is called the stages space. Let us remark that imposing to S to be finite, means in fact imposing a restriction to the stock we are concerned in, for example, a volume restriction.

And it seems to be a normal restriction, if we think at a concrete items store.

Let us now introduce the probabilities $p(i)$, $i \in S$ which form the initial distribution.

We can compute the transition probability $p(i, j)$, $i, j \in S$ from (1.1) using (1.2):

$$p(i, j) = \sum_n P_{i+Q_n^{ij}-j} \tag{1.6}$$

where Q_n^{ij} is Q_n corresponding to $i, j \in S$, $Q_\eta^{ij} \neq Q_\nu^{ij}$, $\eta \neq \nu$, and the sequence of random variables $\{I_n\}_{n \in N}$ becomes a finite Markov chain with the stages space S, initial distribution and transition matrix previously defined.

Moreover $\{I_n\}_{n \in N}$ forms a homogeneous Markov chain, so we can write:

$$\begin{aligned} &P(I_n = i_n \mid I_{n-1} = i_{n-1}, \ldots, I_0 = i_0) = \\ &= P(I_n = i_n \mid I_{n-1} = i_{n-1}) = p(i_{n-1}, i_n) \\ &\qquad i_0, i_1, \ldots, i_n \in S. \end{aligned} \tag{1.7}$$

2. OPERATIONAL CHARACTERISTICS

Let us suppose that the evolution of the stock system we are concerned in, is considerablely disturbed by an accident of replenishment.

We should like to know how the stock manager would take this accident into account in taking decisions rules during later replenishments.

Concerning the sequence $\{I_n\}_{n \in N}$, Markov chain, let A be the accidental event we have just talked about.

Obviously, A is a previous event to the present moment $n > 0$, so we can use a property of Markov chains concerning previous events, that is an extension of Markov Property [3].

Translated in the words of a stock manager this property makes us understand that the policy we shall adopt in following periods must not be influenced by such accidents.

To strengthen this assertion, let us introduce a possible event B, which is subsequent refering to the present, of the same accidental nature as the previous event A.

We want to know if we have to clear the way by adopting a special decision rule.

But another extension of Markov Property refering to the subsequent events says that the possible occurence of B in future, must not be connected with the previous event A.

We have introduced the initial distribution $p(i)$, $i \in S$ and the transition probabilities $p(i, j)$, $i, j \in S$ which describes the chance of transition from the stage i to stage j, after one step.

Using these probabilities, one may define $p(n, i, j)$, the transition probabilities after n steps, by Chapman Kolmogorov relationship:

$$p(n, i, j) = \sum_{k \in S} p(n-1, i, k) p(k, j), \qquad i, j \in S \tag{2.1}$$

where $p(1, k, j) = p(k, j)$, element of the transition matrix joined to the Markov chain.

One can observe that $p(n, i, j)$, is the corresponding element to $p(i, j)$ from the n^{th} power of the transition matrix.

It is useful to compute the probability that the stock level takes a certain important value after a given number of steps.

We can compute all these probabilities using (2.1) or the relationship (1.4) from the previous section.

Thus we might be interested in computing the probability that the stock becomes o after n steps, starting from the stage i, that is:

$$p(n, i, 0) = P(I_{\nu+n} = 0 \mid I_\nu = i), \forall\, \nu > 0. \tag{2.2}$$

But the remark (1-4) involves:

$$p(k_{\eta-1}, k_{\eta-2}) = \frac{P(I_{\eta-2} = k_{\eta-2})}{P(I_{\eta-1} = k_{\eta-1}),}\ , \forall\, \eta > 0. \tag{2.3}$$

In this case:

$$p(n, i, 0) = P(I_{\nu+n} = 0 \mid I_\nu = i) = \frac{P(I_{\nu+n} = 0)}{P(I_\nu = i)}\ , \forall\, \nu > 0\ . \tag{2.4}$$

This is a probability indeed because $A \subset B$ involves $P(A) < P(B)$.

In the same way one can compute the probability that the stock reaches the security level after n steps:

$$p(n, i, s) = P(I_{\nu+n} = s \mid I_\nu = i) = \frac{P(I_{\nu+n} = s)}{P(I_\nu = i)} \tag{2.5}$$

where s *is the stage corresponding to the security level.*

As a matter of fact, $p(n, i, s)$ is the probability that a replenishment is ordered until the moment when the stock reaches his security level starting from the stage i.

In the case when the order and replenishment are not simultaneous, in other words, when the delay or lead time is not zero, we must command before the stock becomes s.

If $p(n, i, s) \geqslant p_s$, where p_s is the given probability for which we begin to take decisions, one may order a replenishment after $n - \tau$ periods, supposing that the reorder time is constant $\tau \geqslant 0$.

Note that for a stock system regarded as a Markov chain the stage space, $S \subset N$.

Telling that the system is in the stage r, means that the inventory contains r items.

So, we can say that S and N have the same order relation.

Let us now compute the probability:

$$P(I_{\nu+n} \leqslant s \mid I_\nu = i)$$

where s is the security level and $\nu > 0$:

$$P(I_{\nu+n} \leqslant s \mid I_\nu = i) = P(\bigcup_{0 \leqslant r \leqslant s} (I_{\nu+n} = r \mid I_\nu = i)) =$$

$$= \sum_{0 \leqslant r \leqslant s} P(I_{\nu+n} = r \mid I_\nu = i), \qquad \forall\, \nu > 0. \tag{2.6}$$

Using the same observation (1.4) we can write:

$$P(I_{\nu+n} \leqslant s \mid I_\nu = i) = \frac{\sum_{0 \leqslant r \leqslant s} P(I_{\nu+n} = r)}{P(I_\nu = i),}, \quad \forall\, \nu > 0. \tag{2.7}$$

Furthermore, we can get from (2.4), (2.5) and (2.7) an interesting relationship between transition probabilities for different horizons.

We shall use the notion of absolute probability of an event, that is the probability that an unconditioned event happens:

$$p(n, i) = P(I_n = i) = \sum_{j \in S} p(j)p(n, j, i)$$

$$n \in N, \qquad i \in S \tag{2.8}$$

where $p(j)$, $j \in S$ is the initial distribution.

Note that the probabilities from the right hand side of (2.4), (2.5) and (2.7) are absolute probabilities.

Replacing by (2-8) we get:

$$P(I_{\nu+n} = 0 \mid I_\nu = i) = p(n, i, 0) = \frac{\sum_{j \in S} p(j)p(n + \nu, j, 0)}{\sum_{j \in S} p(j)p(\nu, j, i)} \tag{2.9}$$

$$P(I_{\nu+n} = s \mid I_\nu = i) = p(n, i, s) = \frac{\sum_{j \in s} p(j)p(n + \nu, j, s)}{\sum_{j \in s} p(j)p(\nu, j, i)} \tag{2.10}$$

$$P(I_{\nu+n} \leqslant s \mid I_\nu = i) = \frac{\sum_{0 \leqslant r \leqslant s} \sum_{j \in S} p(j)p(n + \nu, j, r)}{\sum_{j \in S} p(j)p(\nu, j, i).}. \tag{2.11}$$

We can see that a transition probability can be written as a rational function of other transition probabilities.

3. A REPLENISHMENT POLICY

We shall adopt below a replenishment strategy for the stock we are concerned in, whose evolution is given by:

$$I_n = I_{n-1} + Q_n - R_n.$$

Remember that s is the security level and τ the lead time. Let R be the cardinal of S, that is the maximum capacity of the inventory and suppose that $\tau = 0$, in other words the command and delivery are simultaneous.

The events which modify the stock level (demand or possibly, replenishment) occur at discrete moments. We also suppose that the demand (formulated in number of items) has the next distribution:

$$P(R_n = q) = p_q, \quad q \geqslant 0, \qquad \sum_q p_q = 1 \text{ and}$$

$$p_q = 0 \qquad \text{for} \qquad q > R - s$$

and R_n are independent random variables.

As we showed above, each time that the stock falls under the security level, we have to command a replenishment.

Let us now choose the next strategy:

— when $I_n \leqslant s$ than $Q = R - s$

— when $s + 1 < I_n < R - s$ command in order to complete the stock until the maximum capacity.

Real situations show us that the security stock must not be greater than the rest of the inventory, that is $R - s > s$.

If I_n is the stock level during period n, before the demand and sometimes, before the replenishment, one can write:

$$I_{n+1} = \begin{cases} max(I_n - R_n, 0), & R - s \leqslant I_n \leqslant R \\ max(R - R_n, 0), & s + i < I_n < s + i + 1 \\ & i = 0, \ldots, R - 2s - 2 \\ max(I_n + R - s - R_n, 0), & I_n \leqslant s. \end{cases}$$

We are now primarily concerned in writting the transition matrix of our Markov chain, using previous mentions and choosed strategy.

Let us write some of the most important transition probabilities:

$p(0, 0)$ — command $R - s$

— demand $R - s$ so $p(0, 0) = p_{R-s}$

$p(0, 1)$ — command $R - s$

— demand $R - s - 1$ so $p(0, 1) = p_{R-s-1}$

$p(0, s)$ — command $R - s$

— demand $R - 2s$ so $p(0, s) = p_{R-s-1}$

$p(0, R - s)$ — command $R - s$

— demand 0 so $p(0, R - s) = p_0$

$p(0, R)$ — impossible so $p(0, R) = 0$

$p(1, 0)$ — command $R - s$

— no demand so $p(1, 0) = 0$

$p(1, 1)$ — command $R - s$

— demand $R - s$ so $p(1, 1) = p_{R-s}$

$p(1, s)$ — command $R - s$

— demand $R - 2s + 1$ so $p(1, s) = p_{R-2s+1}$

$p(1, R - s)$ — command $R - s$

— demand 1 so $p(1, R - s) = p_1$.

We get finally, the next transition matrix for the considered Markov chain:

$$
\begin{array}{c}
 \\
0 \\
1 \\
\vdots \\
s \\
s+1 \\
\vdots \\
R-s \\
\vdots \\
R
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & \dots & s & s+1 & \dots & R-s & \dots & R \end{array} \\
\left[\begin{array}{ccccccccc}
p_{R-s} & p_{R-s-1} & \cdots & p_{R-2s} & p_{R-2s-1} & \cdots & p_0 & \dots & 0 \\
0 & p_{R-s} & \cdots & p_{R-2s+1} & p_{R-2s} & \cdots & p_1 & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & \dots & p_{R-s} & p_{R-s-1} & \cdots & p_s & \cdots & p_0 \\
0 & 0 & \dots & p_{R-s} & p_{R-s-1} & \cdots & p_s & \cdots & p_0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
p_{R-s} & p_{R-s-1} & \cdots & p_{R-2s} & p_{R-2s-1} & \cdots & p_0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & 0 & \dots & p_{R-s} & p_{R-s-1} & \cdots & p_s & \cdots & p_0
\end{array}\right]
\end{array}
$$

We can remark from this transition matrix that the studied Markov chain has no transient stages so the whole space S is a closed class.

There is a method [3] we can apply to determine the period and cyclical classes.

In our case the period is 1 and the only cyclical class is the stages space, S.

We can see that the period is 1, looking at the matrix diagonal, whose elements are not zero.

REFERENCES

1. HANSMANN F.: *Operations Research in Production and Inventory Control.* John Wiley and Sons, Inc; New-York, London.
2. HAYES R. H.: *Statistical Estimates Problems in Inventory Control.* Management Science, vol. 15, no. 11, July, 1969.
3. IOSIFESCU M.: *Lanțuri Markov finite și aplicații* (Finite Markov Chains and Applications). Editura tehnică, Bucharest, 1977.

D'UN MODELE DE STOCKS MARKOV

L'auteur présente un schéma aléatoire général à l'aide duquel on met en modèle les situations analysées au chapitre concernant la théorie des stocks dans les recherches opérationnelles.

Cette variante présente des résultats opérationnels intéressants du point de vue des applications.

ОБ ОДНОЙ МАРКОВСКОЙ МОДЕЛИ УПРАВЛЕНИЯ ЗАПАСАМИ

В работе приведена общая алеаторная схема, моделирующая случаи, анализируемые в работах по исследованию операций, в главах, позвящённых теории управления запасами. Изложенный автором вариант содержит интересные с точки зрения приложений результаты.

10. XI. 1977