V. DUMITRU
FLORICA LUBAN
T. DUMITRESCU

Department of Economic Cybernetics
Academy of Economic Studies

Solving MER Optimal Control Problem by SUMT

The paper presents results regarding the efficiency of solving the minimum energy regulator optimal control problem by two new penalty functions, i.e. those proposed by Kaplan [12] and Nguyen-Strodiot [9].

1. THE OPTIMAL CONTROL PROBLEM

The optimal control problem is defined as follows:

Let us find control $u(t) \in U$ such that

$x(t) = f(x(t), u(t), t)$, with the boundary conditions that $x(t_0) = x_0$, $x(t_f) = x_f$ and $x(t) \in X$ and minimizing the objective function (performance index):

$$J(x, u, t) = \Phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt \text{ and}$$

with additional constraints: $g(x(t), u(t)) \leqslant 0$, $h(x(t), u(t)) = 0$

We mention, as particular forms of J:

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 \text{ — minimal time problem,}$$

$$J = \int_{t_0}^{t_f} \|u(t)\|\, dt \text{ — minimal effort problem}$$

$$J = \int_{t_0}^{t_f} u^2(t)\, dt \text{ — minimal energy problem.}$$

2. THE MINIMUM ENERGY REGULATOR PROBLEM

J. C. Descheneau [4] studies the solving of the following optimal control problems concerning the minimal energy regulator proposed by J. F. Hauer in [8]:

$$min\ J = \int_{t_0}^{t_f} u^2(t)\mathrm{d}t$$

such that:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

and with the boundary conditions:

$$x_1(0) = -1 \qquad x_1(t_f) = 0$$

$$x_2(0) = 0 \qquad x_2(t_f) = 0$$

$$t_0 = 0 \qquad t_f = 2.20$$

ubjected to the following constraints:

$$g(x,\ t) = x_1 + 3x_2 + 0 \cdot 02t - 1 \cdot 5 \leqslant 0; -1 \leqslant u \leqslant 1.$$

3. ONE MATHEMATICAL PROGRAMMING APPROACH

In discretizing the problem with a first order approximation and by dividing the $[t_0,\ t_f)]$ into $m = 30$ equal intervals of length Δt, Descheneau reduces this problem to a quadratic programming problem, with 92 constraints and 30 variables:

$$min\ J = \Delta t \sum_{i=0}^{m-1} u^2(i)$$

with conditions:

$$\left.\begin{aligned} u(i) - 1 &\leqslant 0 \\ -u(i) - 1 &\leqslant 0 \end{aligned}\right\} \ i = 0,1\ ...,\ m-1$$

$$\sum_{k=0}^{-1}[(\Delta t)^2\ (i - k - 1) + 3\ \Delta t]\ u(k) + 0 \cdot 02\ (i\Delta t) - 2 \cdot 5 \leqslant 0,\quad i = 1 \qquad m$$

$$-(\Delta)^2 \sum_{k=0}^{m-2} (m - k - 1)\ u(k) + 1 \leqslant 0; -\Delta t \sum_{k=0}^{m-1} u(k) \leqslant 0$$

the last two equations representing the discretizing of conditions $x_1(m) = 0$ and $x_2(m) = 0$. Although of not too great concern in what follows, we consider that Descheneau's equivalence of the conditions $x_1(m) = 0$ and $x_2(m) = 0$ only with the last relation of the above model is not valid as they are not

automatically satisfied. Consequently, instead of the last two inequalities of Descheneau's model, which we will denominate M_0, we considered:

$$-\varepsilon^2 \leqslant \Delta t^2 \sum_{k=0}^{m-2} (m-k-1)\, u(k) - 1 \leqslant \varepsilon^2$$

$$-\varepsilon^2 \leqslant \Delta t \sum_{k=0}^{m-1} u(k) \leqslant \varepsilon^2$$

i.e. a model, whom we shall denominate M_1, with 94 constraints and 30 variables. If the optimizing technique needs the knowledge of an interior point, then as in [4], we considered a version M_{12} of model M_1, viz.:

$$min\ J = \Delta t \sum_{i=0}^{m-1} u^2(i) + Ms^2$$

subjected to:

$$u(i) - 1 \leqslant 0, \qquad i = 0, 1, \ldots, m-1;$$

$$-u(i) - 1 \leqslant 0, \qquad i = 0, 1, \ldots, m-1.$$

$$\sum_{k=0}^{i-1} [(\Delta t)^2 (i-k-1) + 3\,\Delta t]\, u(k) + 0 \cdot 02\,(i\Delta t) - 2 \cdot 5 \leqslant 0,\ i = 1, \ldots, \mathrm{m}$$

$$-s^2 \leqslant \Delta t^2 \sum_{k=0}^{m-1} (m-k-1)\, u(k) - 1 \leqslant s^2$$

$$-s^2 \leqslant \Delta t \sum_{k=0}^{m-1} u(k) \leqslant s^2$$

i.e. a model with 94 constraints and 31 variables.

Finally we studied the efficiency of solving the problem for model M_2, equivalent of M_1, wherein the first two conditions

$$u(i) - 1 < 0, \qquad i = 0, 1, \ldots, m-1;$$

$$-u(i) - 1 < 0, \qquad i = 0, 1, \ldots, m-1.$$

were replaced by conditions $[u(i)]^2 - 1 \leqslant 0$, $i = 0, \ldots, m-1$, i.e. a model with 64 constraints, among which some are quadratic instead of linear, and 30 variables, as well as model M_{22}, equivalent to M_2, but analogous to M_{12}, wherein after introducing the artificial variables, with a coefficient of penalty big enough, we succeed in finding more easily an interior point within the extended admissibility domain. Model M_{22} has 64 constraints and 31 variables.

In [4], Descheneau presents some efficiency results for solving model M_{01}, i.e. model M_0 to which the artificial variables was added both in the case of projected gradient method (GPM) [12] and in SUMT technique solving [7]. For both methods, Descheneau uses as optimizing algorithms without constraints the following ones: Cauchy's gradient algorithm (C), parallel tangents algorithm (PARTAN), conjugated directions algorithm (CD) and

Davidon Fletcher-Powell algorithm (DFP). As concerns the penalty with the SUMT method, the Parissot's logarithmic penalty is used.

4. NUMERICAL RESULTS GIVEN IN [4]

The efficiency results given by Descheneau in [4] are summarized in the following table:

Table 1

Algorithm Method	C	PARTAN	CD	DFP	Efficiency components
GP	30 39 16.589	25 29 14.822	24 27 14.604	24 27 20.231	#IT #NF TIME
SUMT	—	138* 848 50.808	> 280* > 1296 > 120.0	159 634 58.759	#IT #NF TIME

Were:

IT represents the iterations number;

NF — total number of function evaluation.

Time — the time in seconds, necessary to solve the problem with the IBM 360/67.

— : the algorithm was not used in solving the problem for the respective case.

*: a result wherein the problem cannot be considered as solved with sufficient accuracy.

The initial penalty parameter, within the SUMT method, was $r_1 = 1$, and $r_{k+1} = r_k/5$.

The paper does not give the numerical figures of the various solutions, but the graphical result of the optimal solution.

From the table it results that the problem was considered as solved, in all cases, by the projected gradient method, and with the SUMT method, only when using, as an optimization algorithm without constraints the DFP algorithm. The computing times, given in the table, seem to be extremely small with respect to the authors' experience.

5. NEW PENALTY FUNCTIONS

The authors of this paper have been interested in obtaining some efficiency results, by using SUMT technique with two new penalty functions: Kaplan's penalty [12] or Nguyen-Strodiot's penalty [9].

The SUMT method with Kaplan penalty essentially consists in the following:
The problem

$$(\text{PI})\begin{cases} min\ f(x) \\ x \in \Omega \subset R^n \mid g^i(x) \leqslant 0\ \{i = 1, ..., m\} = I \\ f \text{ and } g^i \text{ convex functions} \end{cases}$$

being given, the solution of (PI) may be obtained through a problems sequence:

$$(\text{PII})\ \{min\ f(x) + \Phi_k^\theta(x) \mid \Phi_k^\theta(x) = \sum_{i \in I} \psi_k(g^i(x))\}$$

where functions ψ_k meet certain conditions given in [12]. As an example of penalty function that fulfills the Kaplan conditions, the following is given: $\Phi_k^\theta(x) = r_k \sum_{i \in I} (g^i(x) + \sqrt{(g^i)^2 - r_k^{-2-\theta}}$, $r_k \to \infty$, when $k \to \infty$. As concerns the parameter θ, it is fixed, but the problem of the most efficient interval within which this can vary for a certain class of problems, remains open. The experimental tests [6] on classical test-problems seem to show as an efficient value $\theta \in \{4, 5\}$.
The optimal control problem we are discussing was solved in case of Kaplan penalty for $= 5$.

As concerns the SUMT method with Nguyen-Strodiot penalty, we have the following algorithm [9]:

Given the problem:
(PI): $min\ f(x)$, with conditions $g_i(x) \leqslant 0$, $i = 1, ..., m$, for which: (i) $f, g_i \in C^2$ in the neighborhood of the optimum point x^*, *(ii)* the strict complementary relation, i.e. $\lambda_i^* > 0$ if $g_i(x^*) = 0$, takes place in point (x^*, λ^*), where λ are the Lagrange multipliers. (iii) (x^*, λ^*) is a Kuhn-Tucker point, satisfying the second order condition of sufficiency, and supposing that it satisfies the regularity conditions for g_i; then the solution of the problem (PI) can be approximated with any accuracy, for a sufficiently big r_k, by the solutions of sequence of the problems without constraints:

$$(\text{PII}):\ min\ F(x, y, r_k) = f(x) + \frac{1}{r_k} \sum_{i=1}^{m} y_i^2(e\,x\,p(r_k g_i\ (x)) - 1),\ y \in R_+^m,\ r_k > 0$$

where $y^0 \in R_+^m$ and $x^0 \in R^0$
so that $\nabla_1 F(x^0, y^0, r_k) = 0$, where ∇_1 denotes the gradient of F with regard to the first argument; and at the sequence k,(x^{k-1}, y^{k-1} being given, x^k is determined, such that $\nabla_1\ F(x^k, y^{k-1}, r_k) = 0$; and $y_i^k = y_i^{k-1}\ exp\ (r_k g_i(x^k)/2)$, $i = 1, ..., m$.
If x^k is not unique, x^k is taken as the closest in norm to x^{k-1}.

6. NUMERICAL RESULTS BY SUMT TECHNIQUE WITH KAPLAN PENALTY AND NGUYEN-STRODIOT PENALTY

Some efficiency results concerning the solving of the optimal control problem through the two penalties are given in tables 2 and 3. We mention that, for the obtained results, $r_1 = 1$ and $r_{k+1} = 4r_k$, were used, and as stop

test $\|\nabla F(x_k^*, r_k)\| < 10^{-6}$ or the maximal number of iterations NMAX $\geqslant$ 50. A graph of the optimal control and trajectory is given in Figs. 1 and 2. Although some previous tests carried out by the authors — which will be published in a further study, shows a greater convergence rate for the exponential penalty, we consider that the presented examples show only the fact that both penalty types can be used for solving as Descheneau also assessed previously, such an optimal control problem only with approximation, but better results can be obtained with special designed for algorithms. If we infer that the Kaplan penalty has a smaller convergence rate, then we must also mention the fact that the exponential penalty produces difficulties by an exponent overflow in the computer. Another result which deserves to be mentioned and be possible studied on more examples is that, that in the SUMT method, quadratic constarints are to be preferred as against linear constraints which are strictly verified by the optimal point. The tests have been carried out with a FORTRAN programme written by the authors on an IBM 370/135 computer.

Table 2

Results obtained by the SUMT method with the DFP algorithm with Kaplan penalty

MODEL		M_1	M_2
r_k	94	constraints, 30 variables	64 constraints, 30 variables
$r_1 = 1$	IT NF $F(x_k^*, r_k)$ $\|\nabla F(x_k^*, r_k)\|$ $f(x_k^*)$		
$r_2 = 4$	IT NF F F ff		
$r_3 = 16$	IT NF F F f		
$r_4 = 24$	IT NF F F f		

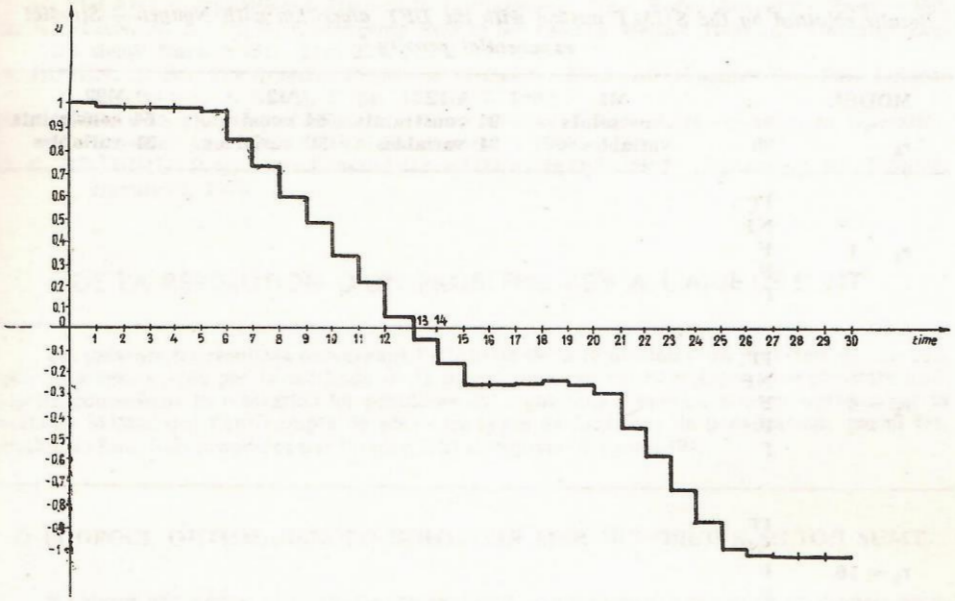
Notes

1. *Initial point:* $u_i^\circ = 0.5$ for $i = 1, ..., 15$; $u_i^\circ = -0.5$ for $i = 16, ..., 30$.

2. *Stop test:* for the M_1 model the maximum number of iterations (NMAX $\geqslant$ 10) was established, or $\|\nabla F(x_k^*, r_k)\| \leqslant 10^{-6}$.

For the M2 model, the stop test was NMAX $\geqslant$ 50 or $\|\nabla F(x_k^*, r_k)\| \leqslant 10^{-6}$.

3. For $r = 4$ in case of the M2 model, the programme was interrupted owing to the exceeding of the given time.


u
1
0.9
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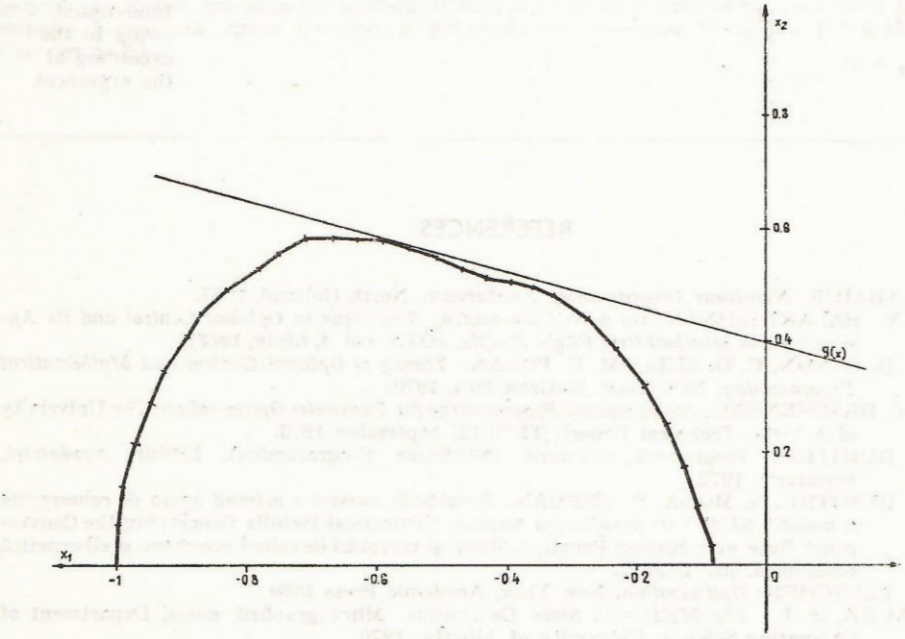
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
time

Fig. 1


x2
0.8
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g(x)
x1
-1
-0.8
-0.6
-0.4
-0.2
0

Fig. 2

Table 3

Results obtained by the SUMT method with the DFP algorithm with Nguyen — Strodiot exponential penalty

MODEL		M1	M12	M2	M22
	94	constraints	94 constraints	64 constraints	64 constraints
r_k	30	variables	31 variables	30 variables	31 variables
$r_1 = 1$	IT NF F F f				
$r_2 = 4$	IT NF F F f				
$r_3 = 16$	IT NF F F f				
$r_4 = 24$	IT NF F F f				Interrupted owing to the exceeding of the argument

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DE LA RESOLUTION D'UN PROBLEME REM A L'AIDE DE SUMT

On présente les résultats concernant l'efficacité de la résolution d'un problème de contrôle optimal à restrictions par la méthode de la programmation mathématique. Les résultats d'efficacité concernent la résolution du problème du régulateur d'énergie minime surtout par la méthode SUMT, qui tient compte de plusieurs types de fonctions de pénalisation, parmi lesquelles les fonctions proposées par Kaplan (12) et Nguyen-Strodiot (9).

О ВОПРОСЕ ОПТИМАЛЬНОГО КОНТРОЛЯ MER ИСПОЛЬЗУЯ МЕТОД SUMT

В работе приведены результаты, касающиеся эффективности решения проблемы оптимального контроля с ограничениями при помощи методов математического программирования. Результаты эффективности касаются решения проблемы регуляторя минимальной энергии, особенно при помощи метода SUMT, в котором учитываются несколько типов штрафных функций, среди которых и функции, предложенные Капланом [12] и Нгуен-Стродётом [9].

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