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The Present State of the Linear Production and Cost Theory

In this paper the general approach of the Leontief production function is described. Based on the input-output-system a multi-part, multi-stage, multiproduct function is derived under real conditions of industrial job-shops. This attempt implies especially movements of material stocks, production losses and distinct types of output-programs. (Programs of products, stored goods, merchandise, work in process, spare parts, in-plant performance.) Finally, the structure of multi-dimensional hyperplanes of costs is characterized.

A. THE BASIC PRINCIPLES OF PRODUCTION AND COST THEORY

1. Subjects of discussion in the production and cost theory

The central point in the production theory is the manufacturing process which deals with the production and realization of the output.

Production theory analyses the quantitative relationships existing between real goods [1] which enter the production process (input) and those which are produced (output).

The theory formulates statements concerning the periodical nature of the input-output interaction within the manufacturing process. The production theory intends to create a system enabling one to explain or forecast the level of output from a given input under realistic conditions.

As inputs are generally scarce, a price determined by the supply and demand ratio can be attributed to the inputs. Costs are defined as the sum of the quantities of goods directly used in production multiplied by their prices. Production costs are a consequence of the input of goods into the economic process.

[1] Real goods are those economic goods which are not money. They include material and invisible physical goods, as well as titles to them.

These production costs, as well as the independent variables which determine them, constitute the subject to be approached in the cost theory. Within the framework of the cost theory an analysis is made of the normal relationship existing between the level of the costs and the independent varibales which determine the costs. From these principles, one should be able to explain or forecast the level of costs depending on changing values of the independent variables. The production theory is an important basis for defining the cost theory because the costs are a consequence of the use of inputs. In the cost theory, the same economic processes are described, as in the production theory, the only difference being that the former adds the price factor.

2. Specific forms of the manufacturing process

A formulation of a system of basics concerning the production theory presupposes an exact characterisation of its subject. In addition, it is compulsory to have a most precise description of the various forms of the economic processes dealing with real goods within the enterprise.

The forms of the manufacturing processes can be defined according to various criteria, for example "the type of manufacturing programme" or to "the type of manufacturing procedure".

The output of the enterprise can be defined as a manufacturing programme. This is characterised by the type and quantity of the products which are produced over a finite amount of time. The first criterion used to characterise the manufacturing programme is "the type of goods". According to this criterion, one can distinguish between material and invisible goods. Many invisible goods can hardly be covered by the production theory, due to their limited capability of being quantified. Therefore, all considerations within the production theory are mostly limited to material goods. According to the *criterion, i.e. "the number of types of goods", one can further distinguish* between a single manufacturing system and a multiproduct manufacturing system.

The typification of products leads to distinctions between mass-production, production of assortments, serial manufacture and single-part production. The manufacturing programme of the enterprise is not comprised only of the customers orders, but also of special partial programmes such as: product sale programme, stored goods programme, merchandise programme, work in process programme, spare parts programme and in-plant performance programme. Finally one can distinguish according to the criterion of "the number of parts of the product" between one part products and multi-parts products.

The quantitative relationships between the input and output of goods within the enterprise depend not only on the type of programme, but also on the type of manufacturing procedure as well. "The number of manufacturing stages", as a criterion, leads to the division of the manufacturing procedures into one stage procedures and multi-stage procedures. According to the criterion of "degree of products divergency" one distinguishes as follows: diverging manufacturing procedures (one input product is split into two or more output products), neither diverging nor converging manufacturing procedures (one single input product enters the production process and one single output product comes out), and converging manufacturing procedures (two or more

input products form one output product). When the manufacturing procedures are described according to the criterion of "the continuity of the production run", one can distinguish between continous manufacturing procedures and discontinous ones. The criterion of "the mechanisation degree" leads to the distinction between manual manufacturing procedures and mechanised ones. According to "the technology used" as a last criterion, one distinguishes between physical-technological, chemical-technological and biological-technological procedures.

The periodicity of the input-output relationship has to reflect as precisely as possible the current situation of the manufacturing process at each given occurrance.

B. THE DESCRIPTION AND ANALYSIS OF THE SYSTEM OF THE BASICS

1. Quantitative structures of the industrial manufacturing processes as a basis for the function on production for empirical use

Observation and analysis of a large part of the industrial manufacturing processes inform us as to specifications necessary for these processes:

(1) Multi-product manufacturing,
(2) Multi-stage manufacturing,
(3) Proportional dependency of the quantity of input on the manufacturing programme,
(4) Constant quantitative proportions of the input combination,
(5) Limited divisibility of inputs.

At the level of industrial enterprise, these conditions prevail in the form of quantity reports specifying the parts produced within the enterprise, parts purchased and standard parts, all of which are necessary for manufacturing a unit of prefabricated or finished products. Such reports are called specification lists. They describe the quantitative structural relationship between raw materials, prefabricated goods and finished products. At the level of setup enterprise, the quantitative relationship between finished products, assemblies of classes of various ranks and components are so characterised that one should immediately be able to represent them in a diagram form. The applied form of the diagram is called "Gozinto diagram" or "production genealogical tree". This is the general form of representation of the specification list decomposition aiming to estimate or plan the required quantity of parts. In the Gozinto diagram, figure 1, the nodes T_1 to T_6 represent components; G_7 and G_8 represent assemblies of lower ranks, G_9 and G_{10} represent assemblies of higher classes and the nodes X_{11} and X_{12} finished products. The figures on arrows identify the number and type of components which form assemblies of lower rank, the number and type of components and lower rank assemblies forming assemblies of higher class, as well as the number and type of components and assemblies of various ranks making up the finished products. Generally, the diagram expresses the input-output relationship within the setup process which has been considered.

According to its structure, it is an open, static, input-output model describing a multi-stage setup process where input is characterised by con-

stant quantitative proportions and its consumption depends directly on the quantity of finished products. The divisibility of input is normally limited to 1.

These interactions are more easily formalised when expressed as a matrix. The input-output square matrix A has an i row which supplies and a j column which receives.

Each element of the matrix a_{ij} is defined as a certain number of pieces which is directly supplied from the supplying row i to the receiving column j.

The direct consumption matrix corresponding to the Gozinto diagram of the figure 1 has the following structure:

$A = (a_{ij}) =$

FROM \ TO	T_1	T_2	T_3	T_4	T_5	T_6	G_7	G_8	G_9	G_{10}	X_{11}	X_{12}
T_1	0	0	0	0	0	0	1	0	2	0	3	0
T_2	0	0	0	0	0	0	0	3	0	2	0	4
T_3	0	0	0	0	0	0	4	0	0	0	0	0
T_4	0	0	0	0	0	0	0	5	2	0	0	0
T_5	0	0	0	0	0	6	6	0	0	3	0	0
T_6	0	0	0	0	0	0	0	0	1	0	0	0
G_7	0	0	0	0	0	0	0	0	1	2	0	0
G_8	0	0	0	0	0	0	0	0	0	3	0	0
G_9	0	0	0	0	0	0	0	0	0	0	1	0
G_{10}	0	0	0	0	0	0	0	0	0	0	0	1
X_{11}	0	0	0	0	0	0	0	0	0	0	0	0
X_{12}	0	0	0	0	0	0	0	0	0	0	0	0

(1)

Furthermore, from the direct-consumption matrix A, one can derive the demand-matrix V where elements v_{ij} indicate the total number of elements of the i row required by a unit of elements of the j column, all technical connexions presented being taken into consideration. V can be determined by:

$$V = (v_{ij}) = (E - A)^{-1} \qquad (2)$$

Should the inverse matrix of the discussed example be formed for the difference between the standard matrix E and the direct-consumption matrix A, then the resulting V is as follows:

$V = (v_{ij}) =$

FROM \ TO	T_1	T_2	T_3	T_4	T_5	T_6	G_7	G_8	G_9	G_{10}	X_{11}	X_{12}
T_1	1	0	0	0	0	0	1	0	3	2	6	2
T_2	0	1	0	0	0	0	0	3	0	11	0	15
T_3	0	0	1	0	0	0	4	0	4	8	4	8
T_4	0	0	0	1	0	0	0	5	2	15	2	15
T_5	0	0	0	0	1	0	6	0	6	15	6	15
T_6	0	0	0	0	0	1	0	1	0	3	0	3
G_7	0	0	0	0	0	0	1	0	1	2	1	2
G_8	0	0	0	0	0	0	0	1	0	3	0	3
G_9	0	0	0	0	0	0	0	0	1	0	1	0
G_{10}	0	0	0	0	0	0	0	0	0	1	0	1
X_{11}	0	0	0	0	0	0	0	0	0	0	1	0
X_{12}	0	0	0	0	0	0	0	0	0	0	0	1

(3)

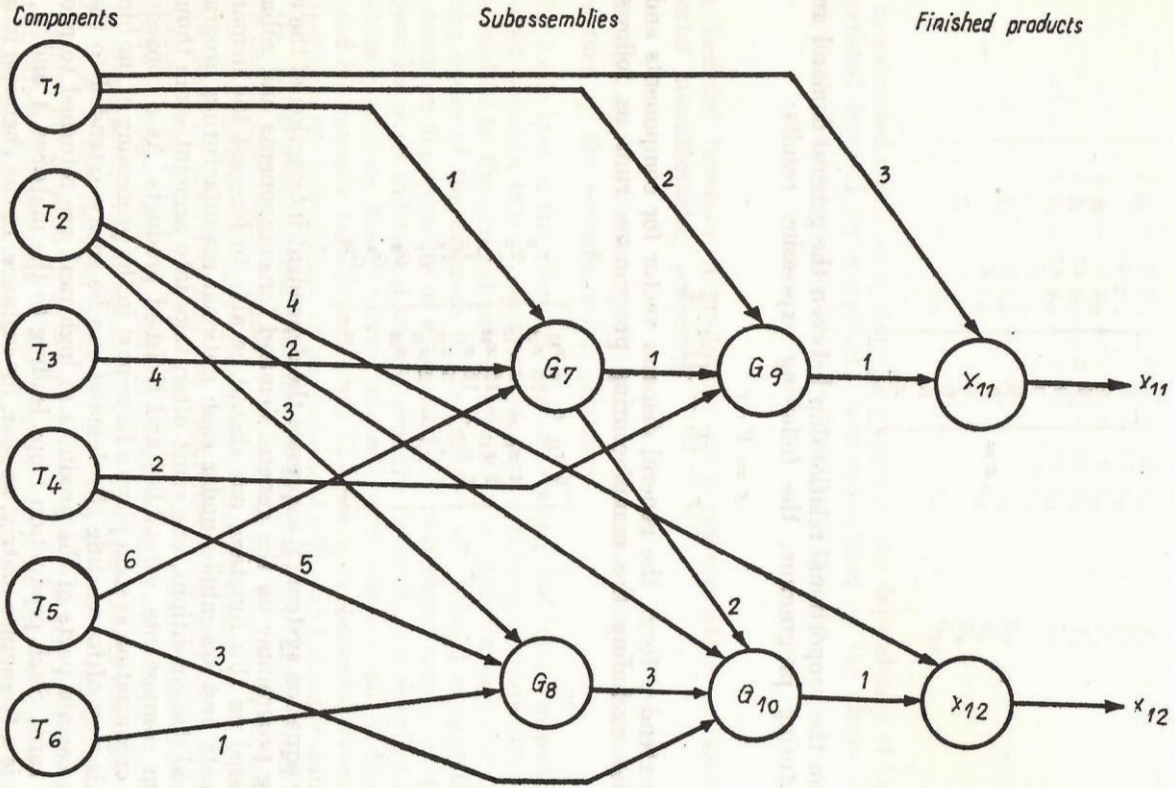


Fig. 1

Gozinto diagram concerning the planning of the demand for parts

When the production structure described by Gozinto diagram of Figure 1 does not change, then V also remains unchanged. The elements of V represent aggregate production coefficients enabling one to forecast the general demand r of components and assemblies under various manufacturing programmes.

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_{11} \\ x_{12} \end{bmatrix}$$

From the proportional relationship between the general demand and the manufacturing programme, the following expression results:

$$r = V \cdot x$$

$$r = (E - A)^{-1} \cdot x \tag{4}$$

In extended form, the general demand vector for components and subassemblies excluding the manufacturing programme runs as follows:

$$r^* = \begin{bmatrix} 6\,x_{11} + 2\,x_{12} \\ 15\,x_{12} \\ 4\,x_{11} + 8\,x_{12} \\ 2\,x_{11} + 15\,x_{12} \\ 6\,x_{11} + 15\,x_{12} \\ 3\,x_{12} \\ x_{11} + 2\,x_{12} \\ 3\,x_{12} \\ x_{11} \\ x_{12} \end{bmatrix} \begin{matrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ G_7 \\ G_8 \\ G_9 \\ G_{10} \end{matrix} \tag{5}$$

The equation system (5) expresses the dependent interaction of the manufacturing programme on the general demand for components and subassemblies. Based on this function, one should be able to forecast the demand for components and assemblies under each optional manufacturing programme. Under realistic conditions, one must often take into account some thousands of various components, assemblies and finished products. As a consequence, the data organisation system plays a large part in the processing of the incurred data. This way of forecasting the demand can be easily extended to the case in which certain parts of the products of high rank are returned to products of lower rank. Such production loops leading to the matrices A and V, none of which is a triangular matrix, appear, for instance, in the chemical industry.

Another extension of the demand forecasting model is that of considerring the changes occurring within the stock of assemblies in order to estimate the periodical demand. In this way, one can plan the production of a surplus

of number of pieces G_9 belonging to the category of subassemblies (spare parts, prefabricated for sale, security stocks) and likewise the reduction in the surplus stock of assemblies G_{10} by g_{10} [1]).

The total demand vector r^* is then correspondingly:

$$r^* = \begin{bmatrix} 3\,g_9 - 2\,g_{10} + 6\,x_{11} + 2\,x_{12} \\ -11\,g_{10} + 15\,x_{12} \\ 4\,g_9 - 8\,g_{10} + 4\,x_{11} + 8\,x_{12} \\ 2\,g_9 - 15\,g_{10} + 2\,x_{11} + 15\,x_{12} \\ 6\,g_9 - 15\,g_{10} + 6\,x_{11} + 15\,x_{12} \\ -3\,g_{10} + 3\,x_{12} \\ g_9 - 2\,g_{10} + x_{11} + 2\,x_{12} \\ -3\,g_{10} + 3\,x_{12} \\ g_9 + x_{11} \\ -g_{10} + x_{12} \end{bmatrix} \begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ G_7 \\ G_8 \\ G_9 \\ G_{10} \end{matrix} \qquad (6)$$

The extended equation system (6) expresses the dependency of the periodical general demand for components and assemblies on the above discussed changes in stocks and on the manufacturing programme.

2. The Leontief function of production as a representative function for industrial manufacturing processes

a) *The derivation of the Leontief production function from the quantitative structure of the manufacturing processes*

The basic idea of the Gozinto diagram stating that input-output relationships exist between two products belonging to various manufacturing stages can be applied to the general case of industrial manufacturing. For that purpose, the nodes of the diagram in the figure 1 are defined as organizational units, meaning departments or areas, and the designated arrows as quantitative flows between individual units (compare figure 2). To simplify matters, production structure has to cover only the procurrement office, the storage dump and the manufacture department. The previous elements corresponding to components are now defined as procurrement office and storage dump from B_1 to B_6. These departments provide the manufacture departaments F_1 to F_{12} with the original input. Further, the departments B_1 and B_6 have to make available two different human labor forces, the departments B_3 and B_4 two different machine activities and the departments B_5 and B_6 two different raw materials. The points F_7 to F_{12} are to represent manufacture departments. To simplify matters again, the marketing departments F_{11} and F_{12} are considered as manufacture departments. These two departments consider exclusively the sale of the finished products X_{11} and X_{12} after fabrication. If no stock changes are allowed in any department, then the output from the procurement offices corresponds to the original input of the enterprise, and the output from the two marketing departments corresponds to the output of

[1]) The level of reduction in the stock is limited by the actual stock level and the structure of the manufacturing programme.

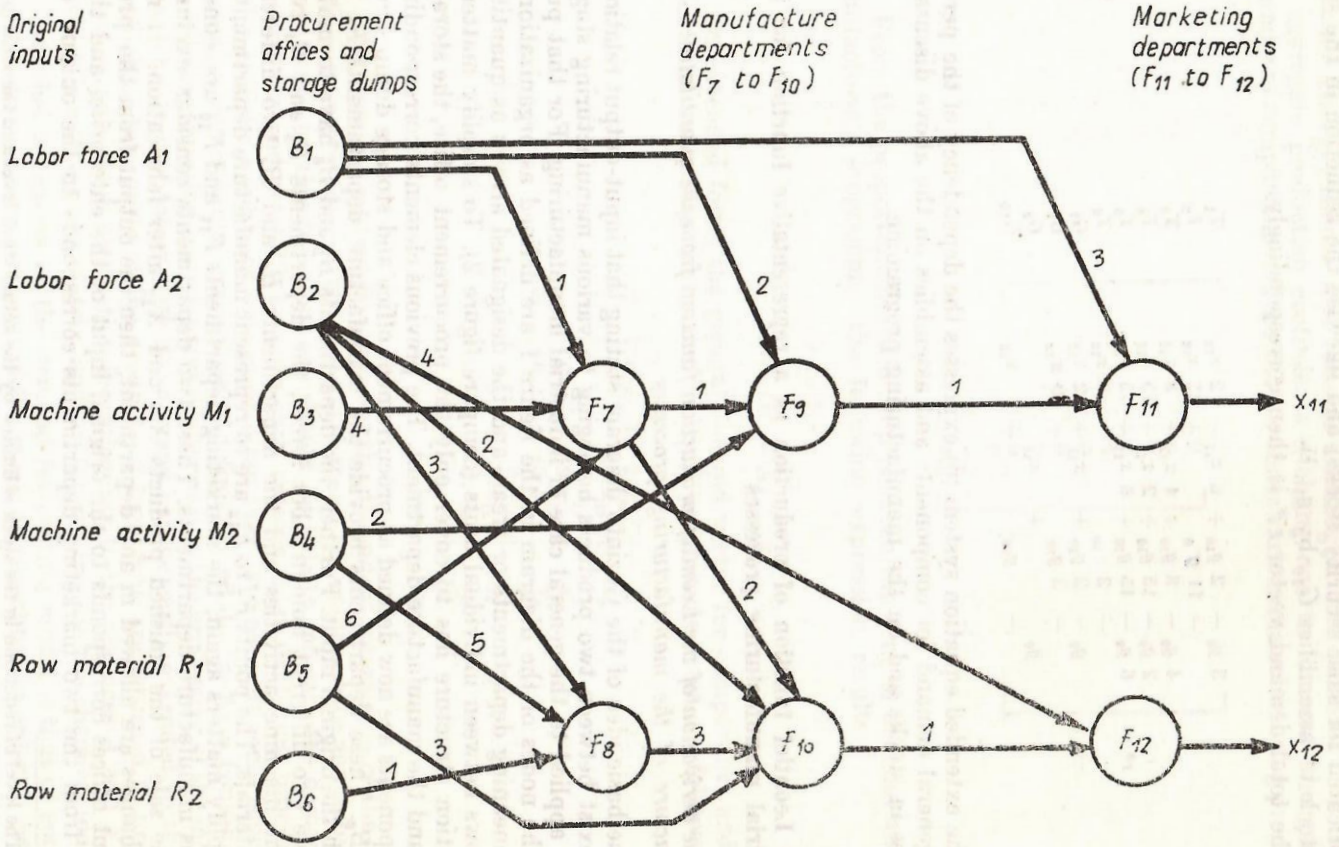


Fig. 2

The production structure of a Leontief production function

the enterprise. For such a production structure, a production function must be formulated as an expression of the quantitative relationship existing between the original input and the output.

The function corresponding to the interrelationships between the quantity of goods r_j either produced or conveyed to the point P_j and the quantity of input r_{ij} procured from the point P_i runs as follows:

$$r_{ij} = a_{ij} \cdot r_j \tag{7}$$

These functions will be called transformation functions. In this situation, a_{ij} returns to the constant production coefficients. Should an intermediate product R_j be produced form two inputs R_{1j} and R_{2j}, then the transformation functions of the department P_j run as follows:

$$\begin{aligned} r_{1j} &= a_{1j} \cdot r_j \\ r_{2j} &= a_{2j} \cdot r_j \end{aligned} \tag{8}$$

Out of this, it is obvious that both inputs are to be always used in constant quantitative ratios.

$$\frac{r_{1j}}{r_{2j}} = \frac{a_{1j}}{a_{2j}} \tag{9}$$

Within a system of $r_{1j} - r_{2j}$ coordinates, these transformation functions are represented by a straight line from the origin of a form such as:

Within a system if $r_1 - r_2$ coordinates, these transformation functions are represented by a straight line from the origin of a form such as:

$$r_{1j} = \frac{a_{1j}}{a_{2j}} \cdot r_{2j}$$

Each point on the straight line reflects a specified quantity of output r_j, corresponding to a production level. Should a quantity of input r_{ij} larger than a_{ij} be used, then a wastage of this input occurs. Input combinations whereby an input is wasted, are called inefficient. Efficient combinations lie, on the contrary, on the indicated straight line. Should one describe the production structure of figure 2 by help of the input-output analysis, then a production system with total number of n production departments results; x_i represents

$$\begin{aligned} r_1 &= a_{11} \cdot r_1 + a_{12} \cdot r_2 + \ldots + a_{1n} \cdot r_n + x_1 \\ &\vdots \\ r_n &= a_{n1} \cdot r_1 + a_{n2} \cdot r_2 + \ldots + a_{nn} \cdot r_n + x_n \end{aligned} \tag{10}$$

The direct consumption matrix A is composed here of constant production coefficients a_{ij}:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} \ldots a_{1n} \\ \vdots \\ a_{n1} \ldots a_{nn} \end{bmatrix} \tag{11}$$

Expressed as a matrix, the production system (10) runs as such:

$$\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} = \begin{bmatrix} a_{11} \dots a_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ a_{n1} \dots a_{nn} \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ r_n \end{bmatrix} + \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \tag{12}$$

$$\text{or} \quad r = A \cdot r + x$$

$$r = (E - A)^{-1} \cdot x \tag{13}$$

Should one assume that the first m rows describe the original input then the first m equations of the above equation system represent the production of the enterprise:

$$r_m = (E - A)_m^{-1} \cdot x \tag{14}$$

At constant production coefficients a_{ij}, this function is a Leontief production function. It will be defined here as a linear production function.

As the direct-consumption matrix A is composed of the constant production coefficients a_{ij}, therefore the consumption matrix

$$V = (E - A)^{-1}$$

comprises only constant elements which are to be defined by v_{ij}. In the case of a multi-product, multi-stage manufacturing enterprise, the Leontief production function consists of the following equation system:

$$\begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_m \end{bmatrix} = \begin{bmatrix} v_{11} \dots v_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ v_{m1} \dots v_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \\ x_m + 1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \tag{15}$$

Proceeding from the consideration that the production structure covers n departments and the first m departments convey the original inputs, then $r_i = r_1$ to r_m are the quantities of original inputs and $x_j = x_1$ to x_n are the quantities of finished products emerging from all manufacturing departments $P_j = P_1$ to P_1 to P_n of the enterprise. As to the manufacture department P_j which does not supply products for sale, x_j is equal to zero. Normally, the initial inputs are not directly marketed. In this case, at least the first m elements of x_j are equal to zero.

As all transformation functions comprise constant production coefficients, then these coefficients constitute the elements of the direct consumption matrix A as well.

	TO	B						F					
FROM		1	2	3	4	5	6	7	8	9	10	11	12
B	1	0	0	0	0	0	0	1	0	2	0	3	0
	2	0	0	0	0	0	0	0	3	0	2	0	4
	3	0	0	0	0	0	0	4	0	0	0	0	0
	4	0	0	0	0	0	0	0	5	2	0	0	0
	5	0	0	0	0	0	0	6	0	0	3	0	0
	6	0	0	0	0	0	0	0	1	0	0	0	0
F	7	0	0	0	0	0	0	0	0	1	2	0	0
	8	0	0	0	0	0	0	0	0	0	3	0	0
	9	0	0	0	0	0	0	0	0	0	0	1	0
	10	0	0	0	0	0	0	0	0	0	0	0	1
	11	0	0	0	0	0	0	0	0	0	0	0	0
	12	0	0	0	0	0	0	0	0	0	0	0	0

(16)

A more profound analysis of the production structure is possible through decomposition of the direct-consumption matrix. Since in the example of figure 1, one only distinguishes between procurrement and manufacture offices, the direct consumption matrix A' will be decomposed into four matrices.

$$A = \begin{bmatrix} A_{BB} & A_{BF} \\ A_{FB} & A_{FF} \end{bmatrix} = \left[\begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\right] \tag{17}$$

The submatrices A_{BB} and A_{FB} are zero matrices in the example considered because no flow of goods occurs between the individual procurrement offices and also none from the manufacture departments to the procurrement offices. The linear production function runs for this reason according to the general statement (14):

$$r = (E - A)^{-1} \cdot x = \begin{bmatrix} E_{BB} & -A_{BF} \\ O_{FB} & (E - A)_{FF} \end{bmatrix}^{-1} \cdot x \tag{18}$$

The production system is for the chosen example as follows:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_{11} \\ r_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 & 6 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 11 & 0 & 15 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 4 & 8 & 4 & 8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 2 & 15 & 2 & 15 \\ 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 & 6 & 15 & 6 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_{11} \\ x_{12} \end{bmatrix} \tag{19}$$

The first six equations of the equation system (19) show the relationship between the initial inputs R_1 to R_6 and the finished products X_{11} and X_{12} They represent the production function of the enterprise:

$$r_m = (E - A)_m^{-1} \cdot x = [E_{BB} \; A_{BF} \cdot (E - A)_{FF}^{-1}] \cdot x \tag{20}$$

or in expanded form for the chosen example such as:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix} = \begin{bmatrix} 6\,x_{11} + 2\,x_{12} \\ 15\,x_{12} \\ 4\,x_{11} + 8\,x_{12} \\ 2\,x_{11} + 15\,x_{12} \\ 6\,x_{11} + 15\,x_{12} \\ 3\,x_{12} \end{bmatrix} \tag{21}$$

Should one compare the Leontief production function for multi-product, multi-stage manufacture (21) with the linear equation system of the demand forecasting according to the Gozinto diagram for setup manufacture (5), then one can recognize that both equation systems have the same structure for the domain of the initial inputs R_1 to R_6. This structure identity also allows the interpretation that the Gozinto diagram, as an instrument of decomposition of the specification list, represents a special form of the Leontief production function for multi-product, multi-stage manufacture. As the forecasting of the demand for parts is a plan estimation, a Leontief production function will be used as a forecasting function. Due to its empirical validity, it allows a precise estimation of the demad for parts, this estimation being dependent upon the manufacturing programme of the respective plan period.

b) *Extensions of the Leontief production function*

In order to be able to explain, or forecast the relationships existing between the use up of original inputs and the output of a planning period under indus-

trial practical operation, one has to extend the derived production function (20) of Leontief Type by two more components. The first component concerns the fact that almost every enterprise works with rejects, the second one states the need of a more differentiated representation of the manufacturing programme (periodical output) in the production function, as before. That is due to the fact that the industrial practical operation does not comprise only the consumers orders, but also achieves storage tasks, in-plant performances, merchandise goods and produces spare parts for consumer service as well. Thus, both the problem of rejects and that of programme differentiation can be easily integrated within the production function.

A reject piece occurs especially at the level of manufacture department. Should each fourth piece be a reject within the manufacture department F_8, then four pieces are to be produced here at each time which enables the other three to pass on to F_{10}. The ratio $\frac{4}{3} = 1{,}33$ is thus the coefficient of rejects c_8 within the department F_8. An analysis of the reject statistics leads to the determination of all reject coefficients c_j belonging to departments j. Should no rejects appear within a department, then the pertinent coefficient is 1. Reject coefficients can be grouped within a reject matrix where only the elements c_{jj} are greater than zero.

$$C = \begin{bmatrix} c_1 & \cdots & 0 \\ 0 & c_2 & \\ \vdots & & \vdots \\ 0 & \cdots & c_{12} \end{bmatrix} = \begin{bmatrix} C_B & 0 \\ & \\ 0 & C_F \end{bmatrix} \tag{22}$$

Should one multiply the direct consumption matrix A on the right side be the reject matrix C, then the quantities of goods actually consumed within the production system result. The function linear in x takes then the form such as:

$$r_m = (E - A \cdot C)_m^{-1} \cdot x$$

$$r_m = [E_{BB} A_{BE} \cdot C_F \cdot (E - A \cdot C_F)_{FF}^{-1} \cdot x \tag{23}$$

The second extension of the Leontief production function concerns the differentiated consideration of the manufacturing programme. In all production functions previously presented [(14), (15), (20), (21) and (23)], the manufacture programme is an independent variable. The vector x is a programme vector comprised of the production quantities of each type of product within a planned period of time. This vector can be conceived as an aggregate term of a number of partial programmes which sum represents the manufacture programme. Besides the programme of products (p), the following programmes are of importance: the programme of stored goods (e), of merchandise (w), work in process (h), spare parts (t), and in-plant performance (i). Although each partial programme refers to a greater extent to certain departments of the production process being considered, the pertinent programme vector can be defined such that the following relationship holds true:

$$x = p + l + w + h + t + i \tag{24}$$

An example for the differentiation of the programme is shown in table 1.

Table 1

Decomposition of the manufacturing programme into partial programmes

Programme of products sale	Programme of storage	Programme of merchandising	Programme of work in process	Programme of spare parts	Programme of in-plant performances	Departments	Criterion
p	l	w	h	t	i		
0	0	0	0	0	0	B_1	Labor force 1
0	0	0	0	0	0	B_2	Labor force 2
0	0	0	0	0	0	B_3	Machine 1
0	0	0	0	0	0	B_4	Machine 2
0	l_5	w_5	0	0	0	B_5	Raw material 1
0	l_6	w_6	0	0	0	B_6	Raw material 2
0	l_7	0	h_7	t_7	0	F_7	Manufacture 7
0	l_8	0	h_8	t_8	0	F_8	Manufacture 8
0	l_9	0	h_9	t_9	i_9	F_9	Manufacture 9
0	l_{10}	0	h_{10}	t_{10}	i_{10}	F_{10}	Manufacture 10
a_{11}	0	0	0	0	0	F_{11}	Marketing department 1
a_{12}	0	0	0	0	0	F_{12}	Marketing department 2

Should one visualize the partial programmes as independent variables of the original input consumption, then the Leontief production function (14) changes to the form (25):

$$r_m = (E - A)_m^{-1} \cdot (a + l + w + h + t + i) \tag{25}$$

or into the decomposed form after taking into consideration the reject coefficients according to (23):

$$r_m = [E_{BB} A_{BF} \cdot C_F \cdot (E - A \cdot C_F)_{FF}^{-1}] \cdot (a + l + w + h + t + i) \tag{26}$$

The decomposition of the manufacture programme to (26) has a great importance for a differentiated planning of the programmes of products, stored goods, merchandise, work in process, spare parts, and in-plant performance.

3. The function of production as a general system of basics for the input-output process

Kloock considers as the central point in his analysis of the production theory the production structure of the enterprise processes. In doing so, he uses the general input-output statement of the production function. He divis-

des the production process into partial processes and production departments. His production function of type D represents a static system of basics and has the manufacture programme of the enterprise as an independent variable. Kloock assumes different transformation functions for the input of labor-force, material potential goods and materials. He distinguishes in this respect, the following types of production departments:

(1) Personnel departments (personnel offices) for supplying the labor force (index A),
(2) Procurrement offices for material potential goods (index M),
(3) Procurrement offices for raw materials, auxilliary materials and power fuels (index S),
(4) Manufacture departments (index F).

The direct consumption matrix A can be divided according to the organisational structure into the following submatrices:

$$A = \begin{bmatrix} A_{AA} & A_{AM} & A_{AS} & A_{AF} \\ A_{MA} & A_{MM} & A_{MS} & A_{MF} \\ A_{SA} & A_{SM} & A_{SS} & A_{SF} \\ A_{FA} & A_{FM} & A_{FS} & A_{FF} \end{bmatrix} \tag{27}$$

With the help of the matrix A (27), one can carry out a clear and precise analysis of the production structure. The type of each respective production structure will become obvious depending on which of the submatrices belonging to the direct-consumption matrix represents a zero matrix. Furthermore, the production structure can be correspondingly characterized depending on whether the direct-consumption matrix takes the form of a triangular matrix, when adequate numbering of the departments. A production structure where relationships can be described by a triangular matrix comprises no loops or cycles. (Kloock refers here to the "simple dependent production systems").

Should one or more production departments within the production flow supply the previous departments by using machines being produced within the enterprise, then cycles or loops do occur. (Kloock refers here to a "complex production system"). The submatrix A_{FM} is, in the case of a non-cyclical production structure, a zero matrix. On the contrary, this does not hold true in the case of a cyclical production structure, i.e. self produced machines.

To formulate the input-output model of production according to the general statement (13)

$$r = (E - A)^{-1} \cdot x$$

it is necessary to decompose the vector r and vector x which correspond to the direct-consumption matrix, into subvectors. For a non-cyclical production structure, the input-output model of the prodution for example runs such as:

$$\begin{bmatrix} r_A \\ r_M \\ r_S \\ r_F \end{bmatrix} = \begin{bmatrix} (E - A)_{AA} & -A_{AM} & -A_{AS} & -A_{AF} \\ 0_{MA} & (E - A)_{MM} & -A_{MS} & -A_{MF} \\ 0_{SA} & 0_{SM} & E_{SS} & -A_{SF} \\ 0_{FA} & 0_{FM} & 0_{FS} & (E - A)_{FF} \end{bmatrix} \cdot \begin{bmatrix} x_A \\ x_M \\ x_S \\ x_F \end{bmatrix} \tag{28}$$

As the submatrices 0_{SA}, 0_{SM}, 0_{FA} and 0_{FM} represent zero matrices in (28), one can determine a partial production function for the input of material consumer goods independent of the input of labor force and potential goods. The following equation system (29) can be obtained for the input of materials, auxilliary materials and power fuels:

$$\begin{bmatrix} r_S \\ r_F \end{bmatrix} = \begin{bmatrix} E_{SS} & -A_{SF} \\ 0_{FS} & (E-A)_{FF} \end{bmatrix}^{-1} \cdot \begin{bmatrix} x_S \\ x_F \end{bmatrix} = \begin{bmatrix} E_{SS} & A_{SF}(E-A)_{FF}^{-1} \\ 0_{FS} & (E-A)_{FF}^{-1} \end{bmatrix} \cdot \begin{bmatrix} x_S \\ x_F \end{bmatrix} \tag{29}$$

From the equation system (29), one obtains the partial production function for the input of materials, auxilliary materials and power fuels:

$$r_S = (E_{SS}A_{SF} \cdot (E-A)_{FF}^{-1}) \cdot \begin{bmatrix} x_S \\ x_F \end{bmatrix} \tag{30}$$

Should one use in the question system (30) the respective transformation function as a basis, then one can obtain the partial production function for materials in the form of a sum. Provided that all the original materials are set in the enterprise (meaning that x_S is a zero vector), then all quantities of goods are dependent on the planned quantities of finished products x_F, on the direct demand A_{SF} of the manufacture departments, and on the indirect demand

$$(E - A_{FF}^{-1})$$

the latter arising from the interlacing between the manufacture departments.

In the same way, one can derive from the input-output model a partial production function for the input of the performance data of the material potential goods. In addition, one can derive according to this method a partial production function for the input of labor force. The latter can be determined only when other two partial production functions have already been determined.

C. *Cost functions based on the multi-dimensional functions of production*

The functions of cost, as well as the multi-variable (multi-dimensional) production functions, on which they are based, can no longer be described as two dimensional systems. From the general statement of the linear production function based on the input-output model, one can derive the cost functions of the variable costs of an enterprise by multiplying the vector $q'_m = (q_1 \text{ to } q_m)$ which represents the price of the inputs, by the vector $r_m = (r_1 \text{ to } r_m)$ representing the original quantities of input:

$$K_v = q'_m \cdot r_m = (q_1 \dots q_m) \cdot \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} \tag{31}$$

Should one use expression (14) for the function of cost (31), then a general statement of an input-output type for the cost function of the variable costs of an enterprise results:

$$K_v = q'_m \cdot r_m = q'_m \cdot (E-A)_m^{-1} \cdot x \tag{32}$$

The elements of the vector q'_m are the prices of inputs or the functional relationships which correspond to the price level. As long as the elements in A and q' are constant coefficients (scalars), (32) has a linear form. The x vector represents the production programme of the enterprise. As this general model covers only the inputs in the field of real goods, it is compulsory to extend it to the inputs of nominal goods [1]) as well, when one wishes to represent the variable costs in the manufacturing and financial field. An adequate extension is necessary in order to integrate the changing cost occurring in the management field, with the cost function. Should one try to formulate a total cost curve, then one has to consider, furthermore, the block of fix costs by estimating the fix quantities of consumed goods $\bar{r}_m$ · A general cost curve based on the input-output statement runs such as:

$$K = q'_m \cdot r_m = q'_m \cdot [(E - A)^{-1}_m \cdot x + \bar{r}_m) \tag{33}$$

Taking into consideration the incurred rejects, (33) changes into the cost function (34), according to the form (23):

$$K = q'_m \cdot \{[E_{BB} A_{BF} \cdot C_F \cdot (E - A \cdot C_F)^{-1}_{FF}] \cdot x + \bar{r}_m\} \tag{34}$$

Should one visualize, according to (24) the particular programmes contained within x, then (34) changes into (35):

$$K = q'_m \cdot \{[E_{BB} A_{BF} \cdot C_F \cdot (E - A \cdot C_F)^{-1}_{FF}] \cdot (p + l + w + h + t + i) + \bar{r}_m\} \tag{35}$$

Should only two independent variables x_1 and x_2 appear in a simplified case, then the cost function (35) describes a hyperplane of costs (Figure 3).

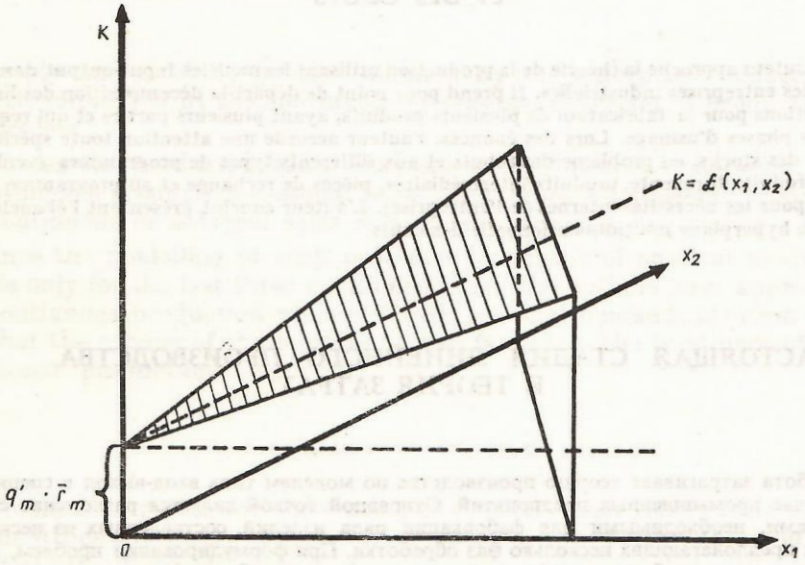


Fig. 3

The linear two dimensional hyperplane of costs.

[1]) Nominal goods are all forms of money as well as titles to money.

In the case of three or more variables, no longer is graphical representation of interactions possible. Accordingly, the corresponding cost curve takes the form of a multi-dimensional hyperplane.

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L'ETAT ACTUEL DE LA THEORIE LINEAIRE DE LA PRODUCTION ET DES COUTS

L'auteur approche la théorie de la production utilisant les modèles input-output dans le cas spécial des entreprises industrielles. Il prend pour point de départ la décomposition des listes des spécifications pour la fabrication de plusieurs produits, ayant plusieurs parties et qui requièrent plusieurs phases d'usinage. Lors des énoncés, l'auteur accorde une attention toute spéciale à la rotation des stocks, au problème des rebuts et aux différents types de programmes (vente, stockage, produits de revente, produits intermédiaires, pièces de rechange et au programme de production pour les nécessités internes de l'entreprise). L'auteur conclut présentant l'ébauche de la structure hyperplane multidimensionnelle des coûts.

НАСТОЯЩАЯ СТАДИЯ ЛИНЕЙНОГО ПРОИЗВОДСТВА И ТЕОРИЯ ЗАТРАТ

Работа затрагивает теорию производства по моделям типа вход-выход в специфическом случае промышленных предприятий. Отправной точкой является разложение списков с реперами, необходимыми для фабрикации ряда изделий, составленных из нескольких частей и предполагающих несколько фаз обработки. При формулировании проблем, особое внимание уделяется обращению запасов, производственному браку и различным типам программ (продажа, запасы, закупленные с целью перепродажи изделия, полуфабрикаты, запасные части и программа производства для внутренних нужд предприятия). В заключени и приведён набросок многомерной гиперплановой структуры затрат.

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