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Some Statistical Considerations Concerning the Reliability Analysis in the Multivariate Case

We consider a system having q components, whose working durations up to failure are random variables $X_1, ..., X_q$, with known distribution.

If not all system's components have simple connections, and X_i, $i = 1, 2, ..., q$ are correlated random variables, the estimation of the system's reliability function must be approached like a problem of multivariate statistical analysis.

In this work, non-parametric estimations will be presented for the reliability function of the system and for the hazard gradient, and some of their properties will be given.

Let $X_1, ..., X_q$, $(X_i > 0, 1 \leqslant i \leqslant q)$ be the working durations, up to failure, of the components of a system. When the distributions of random variables X_i (denoted by $F_i(x)$, $1 \leqslant i \leqslant q$) are known, one can estimate the reliability functions $\bar{F}_i(x) = 1 - F_i(x)$ of the components, and if the system's components connected in series, in parallel or in a bridge, and X_i are stochastically independent, then one can calculate the system's reliability function $\bar{F}(t) = P(X_1 > t, ..., X_q > t)$ in terms of $\bar{F}_i(t)$. If there are components which have not such simple connections, and X_i are correlated random variables, then the estimation of function $\bar{F}(t)$ (or more generally of function $\bar{F}(\mathbf{x})$, $\mathbf{x} = (x_1, ..., x_q)'$) must be approached like a problem of multivariate statistical analysis. We shall assume that the observations on variables X_i are *complete*, in the sense that they represent failure durations.

In the present work, non-parametric estimations will be presented for the system's reliability function $\bar{F}(\mathbf{x})$, $\mathbf{x} \in R^q$ and for the hazard gradient, and some of their properties will be given.

1. NON-PARAMETRIC ESTIMATIONS OF THE RELIABILITY FUNCTION.

Let $\mathbf{X} = (X_1, ..., X_q)'$ be the working durations vector $(\mathbf{X} \geqslant \mathbf{0})$ and $\mathbf{X}_1, ..., \mathbf{X}_n$ a sample on $\mathbf{X}$. Then an estimation of $\bar{F}(\mathbf{x}) = P(X_1 > x_1, ..., X_q > x_q)$ is

$$\bar{F}_n(\mathbf{x}) = \frac{N(\mathbf{x})}{n} \tag{1.1}$$

where $N(\mathbf{x})$ = number of vectors $\mathbf{X}_\alpha$ greater than $\mathbf{x}$, $1 \leqslant \alpha \leqslant n$. We know [6] that

$$E[\bar{F}_n^*(\mathbf{x})] = \bar{F}(\mathbf{x})$$

$$\text{Var}\,[\bar{F}_n^*(\mathbf{x})] = \frac{\bar{F}(\mathbf{x})(1 - \bar{F}(\mathbf{x}))}{n} \leqslant \frac{1}{4n} \tag{1.2}$$

hence $\bar{F}_n^*(\mathbf{x})$ is an unbiased and consistent estimation of $\bar{F}(\mathbf{x})$, if the function $\bar{F}(\mathbf{x})$ has a probability density function $f(\mathbf{x})$, then:

$$\bar{F}(\mathbf{x}) = \int_{\mathbf{x}}^{\infty} f(\mathbf{u})\mathrm{d}\mathbf{u}. \tag{1.1'}$$

Further on, we shall introduce an estimation of $\bar{F}(\mathbf{x})$, using an estimation for $f(\mathbf{u})$. To this end, we shall introduce firstly some definitions [5].

Definition *1.* Let $\{\delta_n\}_{n \in N}$ be a sequence of functions defined on R^q, integrables, which satisfy conditions.

a) For any n and $D \subset R^q$, there is a constant A which does not depend on n (but which can possibly depend on D) such as

$$\int_D |\delta_n(\mathbf{x})|\,\mathrm{d}\mathbf{x} < A. \tag{1.3}$$

b) For any $D \subset R^q$ and $D \ni 0$ (0 is the zero vector from R^q) we have

$$\lim_{n \to \infty} \int_D \delta_n(\mathbf{x})\mathrm{d}\mathbf{x} = 1$$

$$\lim_{n \to \infty} \int_{\bar{D}} |\delta_n(\mathbf{x})|\,\mathrm{d}\mathbf{x} = 0, \quad \bar{D} = R^q/D. \tag{1.4}$$

c) $\delta_n(\mathbf{x}) \to 0$ when $n \to \infty$, uniformly, for any $\mathbf{x} \in \bar{D}$.

The sequence $\{\delta_n\}_{n \in N}$ *is called a sequence of* δ*-functions* [5].

In the preceding definition and in the further down considerations we agree to denote, abbreviatelly

$$\int_{\mathbf{x}}^{\infty} h(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{x_1}^{\infty} \cdots \int_{x_q}^{\infty} h(u_1, ..., u_q)\, \mathrm{d}u_1 ... \mathrm{d}u_q, \quad \mathbf{x} \in R^q$$

or

$$\int_{-\infty}^{+\infty} h(\mathbf{u})\, \mathrm{d}\mathbf{u} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(u_1, ..., u_q)\, \mathrm{d}u_1 ... \mathrm{d}u_q$$

for any integrable function h.

The properties of δ — sequences are studied in [5]. There is a wide class of such sequences, which can be defined by means of *weight functions*. Such a *weight function* is a probability density $K(\mathbf{x})$, $\mathbf{x} \in R^q$ which satisfies conditions

$$1^0.\ K(\mathbf{x}) < \infty, \quad \mathbf{x} \in R^q$$

$$2^0.\ \lim_{||x|| \to \infty} |x_1 \dots x_q|\, K(\mathbf{x}) = 0.$$

The δ-sequence is defined by relation

$$\delta_n(\mathbf{x}) = \frac{1}{h_1(n) \dots h_q(n)} K\left(\frac{x_1}{h_1(n)}, \dots, \frac{x_q}{h_q(n)}\right) =$$

$$= \frac{1}{h_1(n) \dots h_q(n)} K\left(\frac{\mathbf{x}}{\mathbf{h}(n)}\right),\ h_i(n) > 0,\quad 1 \leqslant i \leqslant q \tag{1.4'}$$

where $\lim_{n \to \infty} ||\, \mathbf{h}(n)\, || = 0$. Examples of weight functions may be

$$K(\mathbf{x}) = \frac{1}{\pi^q} \prod_{i=1}^{q} \frac{1}{1 + x_i^2} \tag{1.5}$$

$$K(\mathbf{x}) = \frac{1}{2^q}\, e^{-\sum_{i=1}^{b} |x_i|}. \tag{1.6}$$

Definition *2.* Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be a sample on random vector $\mathbf{X}$. We call estimation of the probability density f in point $\mathbf{x}$

$$f_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \delta_n(\mathbf{x} - \mathbf{X}_j). \tag{1.7}$$

Estimation $f_n(x)$ satisfies the condition

$$\lim_{n \to \infty} E[f_n(\mathbf{x})] = f(\mathbf{x})$$

in any point $\mathbf{x}$ of continuity, of $f(\mathbf{x})$, i.e. it is asymptotically unbiased [5].

$$\text{If}\quad a_n = \int_{R^q} \delta_n^2(\mathbf{x})\, d\mathbf{x} \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n}{n} = 0$$

then $f_n(\mathbf{x})$ defined by (1.7) is a *consistent* estimation for $f(\mathbf{x})$. The following theorem [5] is valid:

Theorem *1.* Weight function $K(\mathbf{x})$ and $\mathbf{h}(n)$ a q — dimensional vector, such as

$$\lim_{n \to \infty} ||\, \mathbf{h}(n)\, || = 0$$

$$\lim_{n \to \infty} n\, h_1(n) \dots h_q(n)\ \infty \tag{1.8}$$

being given, then $f_n(\mathbf{x})$ defined by (1.7) with $\delta_n(\mathbf{x})$ defined by (1.4) is an estimation asymptotically unbiased and consistent for $f(\mathbf{x})$ in any point $\mathbf{x}$ of continuity of $f(\mathbf{x})$. The vector $\mathbf{h}(n)$ can have components of the form:

$$h_i(n) = n^{-\alpha_i}, \quad 0 < \alpha_i < \frac{1}{q}, \quad 1 \leqslant i \leqslant q. \tag{1.8'}$$

On certain conditions, otherwise rather general, we can show [5] that $f_n(\mathbf{x})$ is also asymptotically normal for $f(\mathbf{x})$ when $\mathbf{x}$ is a point of continuity of f.

Taking into account (1.1'), it results that the estimation of reliability function $\bar{F}(\mathbf{x})$ can be defined by relation

$$\bar{F}_n(\mathbf{x}) = \int_{\mathbf{x}}^{\infty} f_n(\mathbf{u})\, d\mathbf{u} \tag{1.9}$$

which will be also written

$$\bar{F}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n} H_n(\mathbf{x} - \mathbf{X}_j) \tag{1.9'}$$

where

$$H_n(\mathbf{x}) = \int_{\mathbf{x}}^{\infty} \delta_n(\mathbf{u})\, du.$$

The following theorem [5] is valid:

Theorem *2.* Estimation (1.9') is an asymptotically unbiased, consistent and asymptotically normal for $\bar{F}(\mathbf{x})$, if $\bar{F}(\mathbf{x}) < 1$. The proof of the theorem relies upon the fact that, for any fixed $\mathbf{x}$ and $\mathbf{y}$

$$\lim_{n\to\infty} H_n(\mathbf{x} - \mathbf{y}) = H(\mathbf{x} - \mathbf{y}) = \begin{cases} 1 \text{ if } \mathbf{x} - \mathbf{y} < 0 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\bar{F}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n} V_{n,j}, \quad V_{n,j} = H_n(\mathbf{x} - \mathbf{X}_j).$$

Hence theorem 2 allows us to construct effectivelly non-parametric estimations, for the multivariate reliability function. If we take the weight functions of the form (1.5) (Cauchy) and (1.6) (absolute-value exponential) and $\mathbf{h}(n)$ a vector with components $h_i(n) = n^{-\alpha_i}$, $0 < \alpha_i < q$ then we shall obtain, respectively, as estimations of the reliability function $\bar{F}(\mathbf{x})$, the following expressions:

$$\bar{F}_n^{(1)}(\mathbf{x}) = \frac{1}{n\pi^q}\sum_{j=1}^{n}\prod_{i=1}^{q}\left[\frac{\pi}{2} - \operatorname{artan} n^{\alpha_i}(x_i - X_{i,j})\right] \tag{1.5'}$$

$$\bar{F}_n^{(2)}(\mathbf{x}) = \frac{1}{n2^q}\sum_{j=1}^{n} e^{-\sum_{i=1}^{q} n^{-\alpha_i}|x_i - X_{i,j}|} = \frac{1}{n2^q}\sum_{j=1}^{n}\prod_{i=1}^{q} e^{-\frac{|x_i - X_{i,j}|}{n^{\alpha_i}}}. \tag{1.6'}$$

Other estimations for $\bar{F}(\mathbf{x})$ may be, of course obtained, when selecting other weight functions.

2. ESTIMATION OF HAZARD GRADIENT

We know, in the case of univariate reliability theory, that $\bar{F}(x) = e^{-\int_0^x r(u)\,du}$ $(F(0) = 0)$ where

$$r(u) = \frac{f(u)}{\bar{F}(u)} = -\frac{d}{du}(\ln \bar{F}(u)) \tag{2.1}$$

is the *hazard rate* or the *failure rate* [1].
Function

$$R(u) = -\ln \bar{F}(u) \tag{2.1'}$$

is called *hazard function.*

In the multivariate case an extension of the notions of *hazard rate* and *hazard function*, is immediately obtained.

Definition *3.* Random vector **X**, with positive components, being given we call *hazard function* [2] the following function:

$$R(\mathbf{x}) = -\ln \bar{F}(\mathbf{x}), \quad \bar{F}(\mathbf{x}) > 0$$

$$\bar{F}(\mathbf{x}) = P(X_1 > x_1, \ldots, X_q > x_q), \tag{2.2}$$

and the q-dimensional vector

$$\mathbf{r}(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} R(\mathbf{x}), \ldots, \frac{\partial}{\partial x_q} R(\mathbf{x})\right) \tag{2.3}$$

is called *hazard gradient.*

When the multivariate reliability function $\bar{F}(\mathbf{x})$, is absolutely continuous, we know [4] that

$$\bar{F}(\mathbf{x}) = \exp\left[-\int_0^x \mathbf{r}(\mathbf{z})\,\mathbf{dz}\right] \tag{2.3'}$$

where, under the integral there is the scalar product

$$\mathbf{r}(\mathbf{z})\,\mathbf{dz} = \sum_{i=1}^{q} r_i(\mathbf{z})\,dz_i$$

and the integral does not depend on the path from **0** to **x**.

Now the problem of estimating the *hazard gradient* function, or more precisely a component $r_i(\mathbf{x})$, $1 \leq i \leq q$, arises.

Before constructing the estimation for the hazard gradient vector, the following notation will be introduced. If **x** is a fixed vector from R^q, we denote by $\mathbf{u}^i$ the $q-$dimensional vector having component $u_i = x_i$. (Hence $\mathbf{u}^i$ is a variable vector from a sub-space of dimension $q-1$; any integral with respect to $\mathbf{u}^{(i)}$ will be hence effected on sub-space of dimension $q-1$).

The following lemma is valid:

Lemma 1. If $F(\mathbf{x})$ allows distribution density $f(\mathbf{x})$, then hazard gradient vector $\mathbf{r}(\mathbf{x})$ has components $r_i(\mathbf{x})$ of the form

$$r_i(\mathbf{x}) = \frac{f_i(\mathbf{x})}{\bar{F}(\mathbf{x})} \tag{2.4}$$

where

$$f_i(\mathbf{x}) = \int_{x^{(i)}}^{\infty} f(\mathbf{u}^{(i)})\, d\mathbf{u}^{(i)}, \quad \mathbf{u}^{(i)} \in R^{q-1}.$$

Proof. Indeed,

$$\bar{F}(\mathbf{x}) = \int_{x}^{\infty} f(\mathbf{u})\, d\mathbf{u} = \int_{x^{(i)}}^{\infty} \left(\int_{x_i}^{\infty} f(\mathbf{u})\, du_i \right) d\mathbf{u}^{(i)}$$

and

$$r_i(\mathbf{x}) = \frac{\partial R}{\partial x_i} = \frac{-\dfrac{\partial \bar{F}}{\partial x_i}}{\bar{F}(\mathbf{x})}.$$

But from the preceding expression of $\bar{F}(\mathbf{x})$ it results

$$\frac{\partial \bar{F}}{\partial x_i} = \int_{x^{(i)}}^{\infty} \frac{\partial}{\partial x_i} \left(\int_{x_i}^{\infty} f(\mathbf{u})\, du_i \right) d\mathbf{u}^{(i)} = \int_{x^{(i)}}^{\infty} f(\mathbf{u}^{(i)})\, d\mathbf{u}^{(i)} = -f_i(\mathbf{x}).$$

Lemma is proved.

We note that, so as to construct the estimation of $r_i(\mathbf{x})$, with fixed i, it is enough to construct an estimation for $f_i(\mathbf{x})$. The estimation of $f_i(\mathbf{x})$ can be, naturally, defined as follows:

Definition 4. The estimation of $f_i(\mathbf{x})$ is

$$f_{i,n}(\mathbf{x}) = \int_{\mathbf{x}^{(i)}}^{\infty} f_n(\mathbf{u}^{(i)})\, d\mathbf{u}^{(i)} \tag{2.5}$$

where $f_n(\mathbf{x})$ is the estimation of density f in point $\mathbf{x}$.

Consequence 1. The δ-sequence $\{\delta_n\}_{n \in N}$ being given, and taking into account (1.7), it results that

$$f_{i,n}(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \delta_n^{(i)}(\mathbf{x} - \mathbf{X}_j) \tag{2.6}$$

where

$$\delta_n^{(i)}(\mathbf{x}) = \int_{\mathbf{x}^{(i)}}^{\infty} \delta_n(\mathbf{u}^{(i)})\, d\mathbf{u}^{(i)}. \tag{2.6'}$$

Proof. From (2.5) we have indeed

$$f_{i,n}(x) = \frac{1}{n} \int_{\mathbf{x}^{(i)}}^{\infty} \delta_n(\mathbf{u}^{(i)} - \mathbf{X}_j)\, d\mathbf{u}^{(i)} = \frac{1}{n} \sum_{j=1}^{n} \int_{\mathbf{x}^{(i)}}^{\infty} \delta_n(\mathbf{u}^{(i)} - \mathbf{X}_j)\, d\mathbf{u}^{(i)}.$$

Making in the last integral the transformation on R^q

$$\mathbf{u}^{(i)} - \mathbf{X}_j = \mathbf{z}^{(i)}$$

we obtain

$$f_{i,n}(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n}\int_{\mathbf{x}^{(i)}-\mathbf{X}_j}^{\infty} \delta_n(\mathbf{z}^{(i)})\, d\mathbf{z}^{(i)} = \frac{1}{n}\sum_{j=1}^{n}\delta_n^{(i)}(\mathbf{x}-\mathbf{X}_j)$$

i.e. (2.6) is proved.

The estimation $f_{i,n}(\mathbf{x})$ has the following property:

Theorem 3. If density $f(\mathbf{x})$ is continuous in any point $\mathbf{x}$, and $|f_i(\mathbf{x})| < \infty$ for any $\mathbf{x}$, if $\delta_n(\mathbf{x}) \geqslant 0$ for any n and any $\mathbf{x}$, and if there is a sequence $\lambda_n^{(i)}$ so that $\delta_n^{(i)}(\mathbf{x}) \leqslant \lambda_n^{(i)}$ and

$$\lim_{n\to\infty}\frac{\lambda_n^{(i)}}{n} = 0 \tag{2.7}$$

then $f_{i,n}(\mathbf{x})$ is an asymptotically unbiased and consistent estimation for $f_i(\mathbf{x})$.

Proof. Indeed,

$$E[f_{i,n}(\mathbf{x})] = \int_{-\infty}^{\infty}\delta_n^{(i)}(\mathbf{x}-\mathbf{y})f(\mathbf{y})\,d\mathbf{y} = \int_{-\infty}^{\infty}\left(\int_{\mathbf{x}^{(i)}-\mathbf{y}}^{\infty}\delta_n(\mathbf{z}^{(i)})\,d\mathbf{z}^{(i)}\right)f(\mathbf{y})\,d\mathbf{y}$$

Making the transformation on R^q

$$\mathbf{z}^{(i)} = \mathbf{u}^{(i)} - \mathbf{y}$$

we have

$$E[f_{i,n}(\mathbf{x})] = \int_{-\infty}^{+\infty}\left(\int_{\mathbf{x}^{(i)}}^{\infty}\delta_n(\mathbf{u}^{(i)}-\mathbf{y})\;d\mathbf{u}^{(i)}\right)f(\mathbf{y})\,d\mathbf{y} =$$

$$= \int_{\mathbf{x}^{(i)}}^{\infty}\left(\int_{-\infty}^{+\infty}\delta_n(\mathbf{u}^{(i)}-\mathbf{y})\,f(\mathbf{y})\,d\mathbf{y}\right)d\mathbf{u}^{(i)}.$$

Hence, as $f(\mathbf{x})$ is continuous, we have, according to theorem 1:

$$\lim_{n\to\infty}E[f_n(\mathbf{x})] = \int_{x^{(i)}}^{\infty}f_i(\mathbf{u}^{(i)})\,d\mathbf{u}^{(i)} = f_i(\mathbf{x}) \tag{2.7'}$$

i.e. $f_{i,n}(\mathbf{x})$ is an asymptotically unbiased estimation for $f_i(\mathbf{x})$. In order to prove that $f_{i,n}(\mathbf{x})$ is a consistent estimation for $f_i(\mathbf{x})$, it is sufficient to prove that its variance vanishes when $n \to \infty$. We have

$$\mathrm{Var}\;[f_{i,n}(\mathbf{x})] = \frac{1}{n}\mathrm{Var}\,[\delta_n^{(i)}(\mathbf{x}-\mathbf{X})].$$

But

$$\mathrm{Var}\,[\delta_n^{(i)}(\mathbf{x}-\mathbf{X})] = \int_{-\infty}^{\infty}\left(\delta_n^{(i)}(\mathbf{x}-\mathbf{y})\right)^2 f(\mathbf{y})\,d\mathbf{y} -$$

$$-\left(\int_{-\infty}^{+\infty}\delta_n^{(i)}(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})d\mathbf{y}\right)^2 \leqslant \lambda_n^{(i)}\int_{-\infty}^{+\infty}\delta_n^{(i)}(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})d\mathbf{y} -$$

$$-\left(\int_{-\infty}^{\infty}\delta_n^{(i)}(\mathbf{x}-\mathbf{y})\,f(\mathbf{y})d\mathbf{y}\right)^2.$$

Hence, taking into account (2.7), (2.7') and the hypothesis that $f_i(\mathbf{x})$ is bounded, it results

$$\lim_{n\to\infty} \mathrm{Var}[f_{i,n}(\mathbf{x})] = 0.$$

The theorem is hence proved.

Notes

1°. If the δ-sequence is defined with the help of a bounded weight function, and $h_i(n)$ is selected of the form (1.8'), then this δ-sequence leads to a sequence $\{\delta_n^{(i)}(\mathbf{x})\}$ which satisfies theorem's 3 conditions.

2°. If $\delta_n^{(i)}(\mathbf{x}) \leqslant \lambda_n^{(i)}$ (condition from theorem 3) and if $\lim\limits_{n\to\infty} \lambda_n^{(i)} = \lambda^{(i)} < \infty$ then

$$a_n^{(i)}(\mathbf{x}) = E\{[\delta_n^{(i)}(\mathbf{x}-\mathbf{X})]^2\} \leqslant \lambda^{(i)2} < \infty$$

$$\gamma_n^{(i)}(\mathbf{x}) = E\{|\delta_n^{(i)}(\mathbf{x}-\mathbf{X})|^{2+\eta}\} \leqslant \lambda^{(i)(2+\eta)} < \infty, (\forall)\eta > 0 \qquad (2.8)$$

in the hypothesis that the previous mean values of $\delta_n^{(i)}(\mathbf{x}-\mathbf{X})$ exist. Hence, in these conditions, it results that the random variables sequence $\{\delta_n^{(i)}(\mathbf{x}-\mathbf{X})\}_{n\in N}$ satisfies the condition

$$\lim_{n\to\infty} n\, E\left[\left|\frac{\delta_n^{(i)}(\mathbf{x}-\mathbf{X}) - E[\delta_n^{(i)}(\mathbf{x}-\mathbf{X})]}{n\,\mathrm{Var}\,(\delta_n^{(i)}(\mathbf{x}-\mathbf{X}))}\right|^{2+\eta}\right] = 0 \qquad (2.8')$$

a fact that can be easily verified. But condition (2.8') is just Lyapunoff's condition [3] for the central limit theorem. Hence the validity of

Theorem *3'*. In theorem's 3 conditions and in conditions (2.8) or (2.8') it results that $f_{i,n}(\mathbf{x})$ is an asymptotically normal estimation for $f_i(\mathbf{x})$.

Conditions (2.8) are fulfilled if weight function $K(\mathbf{x})$ is bounded. Let us now consider, for a fixed i, the component $r_i(\mathbf{x})$ of the haza d gradient.

Definition *5*. The estimation of $r_i(\mathbf{x})$ is

$$r_{i,n}(\mathbf{x}) = \frac{f_{i,n}(\mathbf{x})}{\overline{F}_n(\mathbf{x}),}, \quad \overline{F}_n(\mathbf{x}) \neq 0. \qquad (2.9)$$

We shall prove the following theorem:

Theorem *4*. If the conditions of theorems 2, 3 and 3' are fulfilled, then $r_{i,n}(\mathbf{x})$ is a consistent and asymptotically unbiased estimation for $r_i(\mathbf{x})$. If, moreover, condition (2.8') and

$$\lim_{n\to\infty}\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}}[E[f_{n,i}(\mathbf{x})] - f_i(\mathbf{x})] = 0$$

$$(\mathrm{P})\ \lim_{n\to\infty}\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}}[\overline{F}_n(\mathbf{x}) - E[\overline{F}_n(\mathbf{x})]] = 0,\ \forall\ \mathbf{x}\in R^q \qquad (2.10)$$

the last limit taking place in probability, then $r_{i,n}(\mathbf{x})$ is an asymptotically normal estimation for $r_i(\mathbf{x})$.

Proof. As $f_{i,n}(\mathbf{x}) \xrightarrow{P} f_i(\mathbf{x})$ and $\bar{F}_n(\mathbf{x}) \xrightarrow{P} \bar{F}(\mathbf{x})$, it results that

$$r_{i,n}(\mathbf{x}) = \frac{f_{i,n}(\mathbf{x})}{\bar{F}_n(\mathbf{x})} \xrightarrow{P} \frac{f_i(\mathbf{x})}{\bar{F}(\mathbf{x})} = r_i(\mathbf{x})$$

hence $r_{i,n}(\mathbf{x})$ is a consistent estimation for $r_i(\mathbf{x})$. As $r_i(\mathbf{x})$ is a constant random variable, it results that

$$\lim_{n\to\infty} E[r_{i,n}(\mathbf{x})] = r_i(\mathbf{x})$$

hence $r_{i,n}(\mathbf{x})$ is an asymptotically unbiased estimation for $r_i(\mathbf{x})$. According to theorem 3, we note that the random variable

$$\xi_n = \frac{f_{i,n}(\mathbf{x}) - E[f_{i,n}(\mathbf{x})]}{\left[\frac{a_n^{(i)}(\mathbf{x})}{n}\right]^{\frac{1}{2}}}$$

converges in distribution towards a normal random variable $N(0, 1)$. As $\bar{F}_n(\mathbf{x})$ is a consistent estimation, it results that

$$\frac{\bar{F}(\mathbf{x})}{\bar{F}_n(\mathbf{x})} \xrightarrow{P} 1$$

and hence

$$\Psi_n = \frac{\bar{F}(\mathbf{x})}{\bar{F}_n(\mathbf{x})}\xi_n = \bar{F}(\mathbf{x})\left\{\frac{n}{a_n^{(i)}(\mathbf{x})}\right\}^{\frac{1}{2}}\left[r_{i,n}(\mathbf{x}) - \frac{E[f_{i,n}(\mathbf{x})}{\bar{F}_n(\mathbf{x})]}\right]$$

has asymptotically the normal distribution $N(0, 1)$.

But we are interested to prove that

$$V_n(\mathbf{x}) = \frac{r_{i,n}(\mathbf{x}) - r_i(\mathbf{x})}{\sqrt{\mathrm{Var}(r_{i,n}(\mathbf{x}))}} \tag{2.11}$$

has asymptotically the distribution $N(0, 1)$. We notice firstly that

$$\mathrm{Var}[r_{i,n}(\mathbf{x})] \sim \frac{a_n^{(i)}(\mathbf{x})}{n\bar{F}(\mathbf{x})}.$$

Hence

$$V_n(\mathbf{x}) = \frac{r_{i,n}(\mathbf{x}) - r_i(\mathbf{x})}{\left\{\frac{a_n^{(i)}(\mathbf{x})}{n\bar{F}(\mathbf{x})}\right\}^{\frac{1}{2}}} = \Psi_n + \bar{F}(\mathbf{x})\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} \frac{E[f_{i,n}(\mathbf{x})]}{\bar{F}_n(\mathbf{x})} -$$

$$-\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} f_i(\mathbf{x}) = \Psi_n + \left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} E[f_{i,n}(\mathbf{x})]\left[\frac{\bar{F}(\mathbf{x})}{\bar{F}_n(\mathbf{x})} - 1\right] +$$

$$+\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} [E[f_{i,n}(\mathbf{x})] - f_i(\mathbf{x})].$$

The last term from the second member of the preceding relation vanishes, according to the first relation (2.10); the second term may be written

$$\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} \frac{E[f_{i,n}(\mathbf{x})]}{\bar{F}_n(\mathbf{x})} [\bar{F}(\mathbf{x}) - E[\bar{F}_n(\mathbf{x})] +$$

$$+\left(\frac{n}{a_n^{(i)}(\mathbf{x})}\right)^{\frac{1}{2}} \frac{E[f_{i,n}(\mathbf{x})]}{\bar{F}_n(\mathbf{x})} [E[\bar{F}_n(\mathbf{x}) - \bar{F}_n(\mathbf{x})].$$

In the previous expression, the first term vanishes (according to theorem 2), and the second one vanishes in probability, according to the second condition (2.10). Hence, in other words, V_n has asymptotically the same distribution as variable Ψ_n, hence a distribution $N(0, 1)$. The proof of the theorem is complete.

The result of theorem 4 may be naturally extended to the whole hazard gradient vector $r(\mathbf{x})$, i.e.:

Theorem 4'. If the conditions of theorems 2, 3, 3' and 4 are satisfied for any i, $1 \leq i \leq q$, then the empirical *hazard gradient*

$$\mathbf{r}_n(\mathbf{x}) = (r_{1,n}(\mathbf{x}), \ldots, r_{q,n}(\mathbf{x}))'$$

has asymptotically a normal, q — dimensional distribution $N(\mathbf{r}(\mathbf{x}), \Gamma)$ where the covariance matrix $\Gamma(\mathbf{x})$ has the elements

$$\gamma_{ij}(\mathbf{x}) = \lim_{n\to\infty} \mathrm{Cov}(\delta_n^{(i)}(\mathbf{x} - \mathbf{X}),\ \delta_n^{(j)}(\mathbf{x} - \mathbf{X})). \tag{2.12}$$

Proof. The covariances from (2.12) exist because of the first conditions (2.8) for $1 < i < q$. As $\mathbf{r}_i(\mathbf{x})$ have asymptotically normal distributions, it results that $\mathbf{r}_n(\mathbf{x})$ has a q-dimensional normal distribution (according to lemma 6 from [5], p. 1226).

We note finally that, if the δ-sequence is constructed by means of weight functions $K(\mathbf{x})$, defined by (1.5) or (1.6), the above theorems' conditions are satisfied.

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QUELQUES CONSIDERATIONS STATISTIQUES CONCERNANT L'ANALYSE DE LA FIABILITE D'UN CAS MULTIDIMENSIONNEL

On considère un système ayant q composantes dont les durées de fonctionnement jusqu'à la chute sont $x_1, ..., x_q$, variables aléatoires à répartitions connues.

Si les composantes du système n'ont pas toutes les liaisons simples, et x_i, $i = 1, 2, ..., q$ sont des variables aléatoires corrélées, l'estimation de la fonction fiabilité du système doit être abordée en tant que problème multidimensionnel d'analyse statistique.

L'auteur présente des estimations non-paramétriques pour la fonction fiabilité du système et pour le gradient d'hasard, précisant certaines de leurs propriétés.

НЕКОТОРЫЕ СТАТИСТИЧЕСКИЕ СООБРАЖЕНИЯ, ОТНОСЯЩИЕСЯ К АНАЛИЗУ УСТОЙЧИВОСТИ В МНОГОРАЗМЕРНОМ СЛУЧАЕ

В статье рассматривается система с q компонентами, имеющимися сроки действия до выхода из действия $x_1, ..., x_q$, являющиеся стохастичными переменными с известным распределением.

Если все компоненты системы не имеют простых связей, а x_i, $i = 1, 2, ..., q$, являются коррелированными стохастичными переменными, то оценку функции устойчивости системы необходимо трактовать как проблему многоразмерного статистического анализа.

В настоящей работе представлены непараметрические оценки для функции устойчивости системы и для азартного градиента, показывая и некоторые их свойства.

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