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On Linear Optimization in Optical Data Processing

This paper approaches some aspects of the linear optimization in the case of the optical data processing systems based on holography. Compared with the methods encountered in literature, the author introduces a new optimizing method by using spline functions.

This method is applied to the transfer characteristic. There are also presented some qualitative aspects concerning the digital computer programming of the described mathematical model.

1. INTRODUCTION

In the optical data processing field , a lot of phenomena are to be encountered which produce nonlinearities [1], one of them being determined on account of signal recording on photosensitive materials. In this paper, we will refer to the optical data processing based on holography.

The medium on which the hologram is recorded spatially stores information about the incident irradiance. The transfer characteristic of the medium, that is, the input-output relationship, plays a major role in the holographic process [2] and, of course, in the optical data processing based on holography.

The primary purpose of the recording medium is to store the information about the incident irradiance in such a way that the object wavefront can later be reconstructed. This wavefront is then used to form an image which we want to optimize. For this purpose the problem is to determine the optimum transfer characteristic of the recording medium (photographic emulsion).

Therefore, one of the major problems in the optical data processing systems is getting conditions of minimizing nonlinear distorsions.

Let's consider the $T - E$ curve (amplitude transmittance versus exposure) which is represented in Figure 1.

The linear region of this curve is limited, therefore, it is necessary to *get the best linearizing conditions (optimal linearization).*

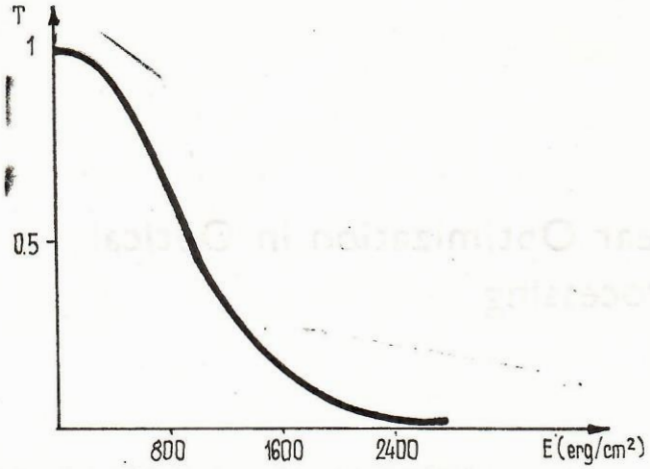

Fig. 1 *Amplitude transmittance versus exposure:* $T = T(E)$

Firstly, we will describe the optimal linearization techniques applied on nonlinear holograms. These techniques optimize in the sense of maximizing the reconstruction of the first order linear hologram output. One of these techniques was described by Francis T.S.Yu [3] and it consists of the following: for a photographic emulsion, the curve represented in Figure 1 is a monotonically decreasing function which can develop in finite power series such as:

$$T(E) = \sum_{n=0}^{N} a_n E^n, \quad \text{for } E \geqslant 0 \tag{1}$$

where T is the amplitude transmittance, E is the exposure, and the a_n's are the real coefficients. The function T, expressed by Eq. (1), is nonlinear.

Let's substitute the nonlinear amplitude transmittance function expressed by Eq. (1), by a linear equation, i.e.:

$$T_1(E) = \lambda_0 + \lambda_1 E. \tag{2}$$

The problem is to choose the parameters λ_0 and λ_1 in order that the Eq. (2) may approximate the Eq. (1), as well as possible.

Let us now consider the difference:

$$T - T_1 = (a_0 - \lambda_0) + (a_1 - \lambda_1) E + a_2 E^2 + a_3 E^3 + \ldots + a_N E^N. \tag{3}$$

To obtain the best condition of approximation it is necessary that:

$$\iint_s (T - T_1)^2 \, dx \, . \, dy = min. \tag{4}$$

In the Eq. (4), s represents the surface of the photographic plate. Also, let us define the spatial ensemble average:

$$<E^n> \triangleq \iint_s E^n(x, y)\, dx \cdot dy; \quad n = 1, 2, 3, ..., N. \tag{5}$$

Then, the optimal choice of the parameters λ_0 and λ_1 can be determined by:

$$\frac{\partial}{\partial \lambda_n} \iint_s (T - T_1)^2\, dx\, .\, dy = 0; \quad n = 0,1. \tag{6}$$

The solution of the Eq. (6) is:

$$\lambda_0 = \frac{\sum_{n=0}^{N} a_n(<E^n><E^2> - <E^{n+1}><E>)}{<E^2> - <E>^2} \tag{7}$$

$$\lambda_1 = \frac{\sum_{n=0}^{N} a_n(<E^{n+1}> - <E^n><E>)}{<E^2> - <E>^2}. \tag{8}$$

The corresponding matrix representation is:

$$\begin{bmatrix} 1 & <E> \\ <E> & <E^2> \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{N} a_n <E^n> \\ \sum_{n=0}^{N} a_n <E^{n+1}> \end{bmatrix}. \tag{9}$$

The Eqs. (7), (8) and (9) offer the good linear approximations with respect to Eq. (2), which is determined by the least-mean-square-error criterion applied to Eq. (4). The Eq. (8) can be considered to be a generalized first order transmittance, meaning the first order transfer function for photographic emulsion. The optimal linearization technique described by Yu tackles the continuous case (on the hologram surface).

Further on, we suggest a linearization technique for the discrete case of the succesive exposures.

2. LINEAR OPTIMIZATION IN THE SUCCESSIVE EXPOSURES CASE

In optical data processing, we meet many cases in which the used holograms are synthetized by successive exposures.

An example in this direction is the case of the high capacity holographic memories synthesis, where the number of successive exposures per recording process — in real time — is of the order of hundreds.

Now let's consider that the exposure E can take k succesive values in the synthesis of the hologram. In this case, the Eqs. (1) and (2) take the following forms:

$$T_{E_i} = \sum_{n=0}^{N} a_n E_i^n; \quad i = 0, 1, 2, ..., k \tag{10}$$

and

$$T_1 = \lambda_0 + \lambda_1 E \tag{11}$$

where T_{E_i} represents the amplitude transmittance value corresponding to the exposure E_i.

To have the best approximation it is necessary that:

$$\sum_{i=0}^{k} (T_{E_i} - \lambda_0 - \lambda_1 E_i)^2 = min. \tag{12}$$

Let's use the following notation:

$$S(\lambda_0, \lambda_1) \triangleq \sum_{i=0}^{k} (T_{E_i} - \lambda_0 - \lambda_1 E_i)^2. \tag{13}$$

Then, to determine the parameters λ_0 and λ_1 we solve the equations:

$$\frac{\partial S(\lambda_0, \lambda_1)}{\partial \lambda_0} = 0 \tag{14}$$

and

$$\frac{\partial S(\lambda_0, \lambda_1)}{\partial \lambda_1} = 0 \cdot \tag{15}$$

Performing the derivatives from Eqs. (14) and (15) we obtain the following results:

$$\sum_{i=0}^{k} (T_{E_i} - \lambda_0 - \lambda_1 E_i)(-1) = 0 \tag{16}$$

and

$$\sum_{i=0}^{k} (T_{E_i} - \lambda_0 - \lambda_1 E_i)(-E_i) = 0. \tag{17}$$

Thus, the Eq. (16) can be written:

$$\sum_{i=0}^{k} T_{E_i} - (k+1)\lambda_0 - \lambda_1 \sum_{i=0}^{k} E_i = 0 \tag{18}$$

or

$$(k+1)\lambda_0 + \lambda_1 \sum_{n=0}^{k} E_i = \sum_{i=0}^{k} T_{E_i} \tag{19}$$

where $(k + 1)$ represents the experimental number of observations about the exposure E_i.

Likewise, the Eq. (17) can be written as:

$$\lambda_0 \sum_{i=0}^{k} E_i + \lambda_1 \sum_{i=0}^{k} E_i^2 = \sum_{i=0}^{k} T_{E_i} E_i. \tag{20}$$

The parameters λ_0 and λ_1 can be obtained from the Eqs. (19) and (20):

$$\lambda_0 = \frac{\sum_{i=0}^{k} T_{E_i} \sum_{i=0}^{k} E_i^2 - \sum_{i=0}^{k} T_{E_i} E_i \sum_{i=0}^{k} E_i}{(k+1)\sum_{i=0}^{k} E_i^2 - \left(\sum_{i=0}^{k} E_i\right)^2} \tag{21}$$

and

$$\lambda_1 = \frac{(k+1)\sum_{i=0}^{k} T_{E_i} E_i - \sum_{i=0}^{k} T_{E_i} \sum_{i=0}^{k} E_i}{(k+1)\sum_{i=0}^{k} E_i^2 - \left(\sum_{i=0}^{k} E_i\right)^2}. \tag{22}$$

Both methods previously described, optimize the curve from Figure 1, in the sense of getting one good approximation.

Although these methods are frequently used, they are not able to specify the behaviour at the extremities of the curve in the sense of the linearization. Therefore, we suggest further on, an approximation method (for the curve from the Figure 1) using linear polynomials on portions (regions), according to the finite element method [4].

3. OPTIMIZATION USING SPLINE FUNCTIONS

Let us now consider that the variable taken into account is the exposure E (see Figure 1). Also, let's consider the following spline functions [4] (particularly, roof functions — see Figure 2) defined on a division of the interval $[0, E_M]$ of the abscissa (the exposure axis; see Figure 1):

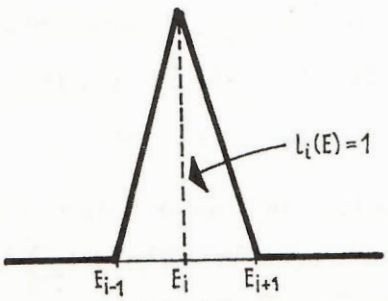

Fig. 2 *Roof function*

$$l_0(E) = \begin{cases} \dfrac{E_1 - E}{E_1}, & \text{for } E \in [0, E_1] \\ 0 \quad , & \text{for } E_1 \leqslant E \leqslant E_M \end{cases} \tag{23}$$

$$l_i(E) = \begin{cases} \dfrac{E - E_{i-1}}{E_i - E_{i-1}}, & \text{for } E_{i-1} \leqslant E \leqslant E_i \\ \dfrac{E_{i+1} - E}{E_{i+1} - E_i}, & \text{for } E_i \leqslant E \leqslant E_{i+1} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{24}$$

and

$$l_{N+1}(E) = \begin{cases} \dfrac{E - E_N}{E_M - E_N}, & \text{for} E_N \leqslant E \leqslant E_M \\ 0 \quad , & \text{for } 0 \leqslant E \leqslant E_N \end{cases} \tag{25}$$

If the division of the segment $[0, E_M]$ is:

$$0 = E_0 < E_1 < E_2 < \ldots < E_{i-1} < E_i < E_{i+1} < \ldots < E_{N+1} = E_M$$

and the step of the division is the same h (see Figure 3), then:

$$l_i(E) = \begin{cases} \dfrac{E - E_{i-1}}{h}, & \text{for } E_{i-1} \leqslant E \leqslant E_i \\ \dfrac{E_{i+1} - E}{h}, & \text{for } E_i \leqslant E \leqslant E_{i+1} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{26}$$

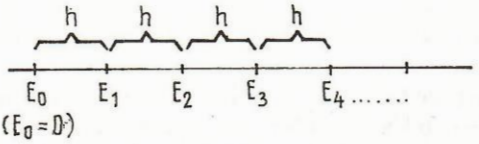

Fig. 3 *The step h of the network is equidistant and can be chosen as small as possible.*

We will consider then, that the network knots are equidistant (see Figure 3).

Let's note the right side of the Eq. (1), thus:

$$f(E) = \sum_{n=0}^{N} a_n E^n . \tag{27}$$

Our problem consists of the approximation of the function $f(E)$ by a linear combination of $l_i(E)$'s functions that is: $\sum_{i=0}^{N+1} \beta_i l_i(E)$, where $l_i(E)$ are functions defined by the Eq. (26).

To approximate the function $f(E)$, using the least-mean-square-error criterion, it is necessary to obtain the coefficients $\beta_0^*, \beta_1^*, \ldots, \beta_{N+1}^*$, so that:

$$\varphi(\beta^*) = \min_{\beta_0,\ldots\beta_{N+1}} \int_0^{E_M} \left[f(E) - \sum_{i=0}^{N+1} \beta_i l_i(E) \right]^2 \cdot \mathrm{d}E \cdot \tag{28}$$

Derivating the Eq. (28) with respect to $\beta_0, \beta_1, \ldots, \beta_{N+1}$, we get the following system of equations:

$$A\beta^* = k \tag{29}$$

where the matrix A is:

$$A = \begin{bmatrix} \int_0^{E_M} l_0^2(E)\,\mathrm{d}E & \int_0^{E_M} l_0(E)\,l_1(E)\,\mathrm{d}E \ldots & \int_0^{E_M} l_0(E)\,l_{N+1}(E)\,\mathrm{d}E \\ & \int_0^{E_M} l_i(E)\,l_j(E)\,\mathrm{d}E & \\ & & \int_0^{E_M} [l_{N+1}(E)]^2\,\mathrm{d}E \end{bmatrix} \tag{30}$$

where the matrix element, a_{ij}, is:

$$a_{ij} = \int_0^{E_M} l_i(E)\,l_j(E)\,\mathrm{d}E \tag{31}$$

and the vector k is given by the expression:

$$\begin{bmatrix} \int_0^{E_M} f(E)\,l_0(E)\,\mathrm{d}E \\ \int_0^{E_M} f(E)\,l_1(E)\,\mathrm{d}E \\ \vdots \\ \int_0^{E_M} f(E) l_{N+1}(E)\,\mathrm{d}E \end{bmatrix} . \tag{32}$$

Taking into account the functions given by the Eqs. (26), the system of equations which must be solved, considering $h = \frac{E_M}{N+1}$, is:

$$\frac{h}{6}\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 4 & 1 \\ 0 & & & 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \vdots \\ \beta_{N+1}^* \end{bmatrix} = \begin{bmatrix} \int_0^{E_M} f(E)\, l_0(E)\, dE \\ \int_0^{E_M} f(E)\, l_1(E)\, dE \\ \vdots \\ \int_0^{E_M} f(E)\, l_{N+1}(E)\, dE \end{bmatrix} \tag{33}$$

On the main diagonal and on the codiagonals, the matrix elements are nonzero and in the rest are zero (i.e., $a_{ij} = 0$, if $|i - j| > 1$). The integrals from the right side of the Eq. (33) can be approximatively calculated, i.e. using the trapez method, and in this case, the system becomes:

$$\frac{1}{6}\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 4 & 1 \\ 0 & & & 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \vdots \\ \beta_{N+1}^* \end{bmatrix} = \begin{bmatrix} f(E_0) \\ \vdots \\ f(E_i) \\ \vdots \\ f(E_{N+1}) \end{bmatrix} \tag{34}$$

where $f(E_i) = \sum_{n=0}^{N} a_n E_i^n$.

The solving techniques of the matrix types are described in literature [5]. The system of equations (34) can be solved by Gauss'method, and we get the solution:

$$f^*(E) = \sum_{i=0}^{N+1} \beta_i^* l_i(E) \tag{35}$$

where $f^*(E)$ represents the approximation of the function $f(E)$ by means of the spline functions.

The discrete case

If $f(E)$ is given only in the points $\{E_i\}_{i=1}^{N}$, then the problem is to minimize the following functional:

$$\Phi(\beta) = \sum_{j=0}^{N} w_j \left[f(E_j) - \sum_{i=1}^{N} \beta_i l_i(E_j) \right]^2 \tag{36}$$

where the weights, w_j, take the following values:

$$w_j = \begin{cases} E_{j+1} - E_j \ , & \text{for } j = 1 \\ E_{j+1} - E_{j-1}, & \text{for } j = 2, .., N-1 \\ E_j - E_{j-1} \ , & \text{for } j = N \end{cases} \tag{37}$$

In the Eq. (36), β is a vector of components $(\beta_1^*, ..., \beta_N^*)$.

4. CONCLUSIONS

The application of the optimal linearization technique expressed by the Eq. (4) means, in fact, solving a continuous case. This technique leads to important conslusions especially regarding the maximization of the reconstruction of the first order hologram image, although a certain degree of nonlinear wavefront recording, on the photographic plate, is admitted. Also, it is possible to get the optimum hologram image reconstruction from a nonlinear holographic recording.

On the other hand, for the noncomplicated signals, optimum nonlinear filters might be obtained, without being restricted to the linear region of the transfer characteristic of the photographic emulsion. The application of this technique at the nonlinear spatial filtering indicates that the signal detection (represented by the first order crosscorrelation) can be spatially separated from the unwanted higher order correlations.

Analysing the Eqs. (7) and (8), one can see that to determine the parameters λ_0 and λ_1, it is necessary to calculate the average $< E^n >$, which is not merely simple, especially when the n's are numerous. In this case computer programming becomes necessary. On the contrary, analysing the Eqs. (21) and (22), for a high number of exposures, it is easier to tackle computer programming, although the precision of the calculus decreases a little.

Regarding approximation of the function $T = T(E)$, using the spline functions, the method presents advantages because it is possible to establish, apriori, the approximation precision, using fixed steps (these steps must be as small as possible). This method offers the possibility of analysing the transfer characteristic on the whole interval [0, 1] of the amplitude transmittance. But we can mark that, for a high precision, some dificulties appear concering the programming, because of the matrix from the left side of the Eq. (33).

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SUR L'OPTIMISATION LINEAIRE DANS LE TRAITEMENT OPTIQUE DES DONNEES

L'auteur présente certains aspects de l'optimisation linéaire dans le cas des systèmes de traitement optique des données utilisant l'holographie.

En comparaison avec les méthodes connues en littérature l'auteur présente une nouvelle méthode d'optimisation à l'aide des fonctions „splines". La méthode est appliquée au cas de la caractéristique de transfert. On présente qualitativement quelques aspects de la programmation du modèle mathématique sur l'ordinateur numérique.

О ЛИНЕЙНОЙ ОПТИМИЗАЦИИ ПРИ ОПТИЧЕСКОЙ ОБРАБОТКЕ ДАННЫХ

В настоящей работе представлены некоторые аспекты линейной оптимизации в случае использования систем оптической обработки данных основанных на холографии.

По сравнению с методами, встречаемыми в литературе, автор приводит новый метод оптимизации, используя сплиновые функции. Метод применяется в случае переходной характеристики. Качественно представлены некоторые аспекты программирования использованной математической модели на Э.В.М.

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