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Some Considerations from the Mathematical Theory of Systems

We present in this paper some topological, probabilistic and informational considerations on some real systems being in interdependence, and we also set the problem of estimating a system's state.

1. GENERALS

As in the systems theory, regardless of their nature, there is a series of senses for the same things, we will give in the following paragraphs some definitions for specifying namely how we further operate.

Definition 1.1. A unitary whole, resulted from the union of several elements of the same kind, is called an ensemble.

Definition 1.2. An ensemble, whose elements are interdependent, constituting as such an organized whole, is called a system.

Definition 1.3. We shall say that a certain part of a system is undivisible if it cannot be considered, or if we do not want it to be considered to be an ensemble of elements.

Definition 1.4. A constitutive element of a system is called a block, if it is indivisible.

Definition 1.5. If the existence of the ensemble is objective, we'll say we have a real system.

Let $\mathcal{S}_1 = \{B_1^i\}_{i \in I}$ and $\mathcal{S}_2 = \{B_2^j\}_{j \in J}$ be two real sytems, whose constitutive elements B_1^i and B_2^j are blocks, and whose nature is not of interest for us, for the time being, and we understand by this that they may be interpreted as concrete or abstract, without precisely stating they represent a certain case.

Let us consider now the real system $\mathcal{S} = \{S_{mn}/m \in I, n \in J\}$ defined by relation

$$S_{mn} = \{\{B_1^{i_k},\}_{i_k=1}^{m}, \{B_2^{j_l}\}_{j_l=1}^{n}\} \tag{1.1}$$

that means that every constitutive element or $\mathcal{S}$ is composed from m blocks of system $\mathcal{S}_1$ and from n blocks of system $\mathcal{S}_2$, the indices i_k j_l evincing that the blocks taken from $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, are arbitrary but with the mention that they are distinct.

Example 1.1. An arbitrary statistic collectivity being given, let us consider the set of characteristics after which it could be studied. The probabilistic distributions of these characteristics depend on certain parameters, as for instance the mean value, dispersion, correlation coefficient, a.s.o. Depending on the considered characteristics number, there can be unidimensional, bidimensional distributions etc., and in each case there can be a certain number of parameters on which the distributions depend.

Taking as $\mathcal{S}_1$ the set of characteristics and as $\mathcal{S}_2$ the set of considered parameters, then $\mathcal{S}$ is just the set of all vectors having at least two components, the one from $\mathcal{S}_1$ and the other from $\mathcal{S}_2$.

Concretely, if we take two characteristics B_1^1 and B_1^2 and five parameters $\{B_2^1\}_{j=1}^{5}$, i.e. the means, dispersions and correlation coefficient between both characteristics, then the elements of $\mathcal{S}$ have the form (B_1^1, B_2^1), (B_1^2, B_2^2), (B_1^1, B_2^1, B_2^3), (B_1^2, B_2^2, B_2^4), $(B_1^1, B_1^2, B_2^1, B_2^2, B_2^3, B_2^4, B_2^5,)$, $(B_1^1, B_2^2, B_2^4,)$ (B_1^1, B_2^5), etc. In the above interpretation, only the first five elements present an interest for us, the other ones do exist theoretically, but do not present any interest; there are totally 93 elements.

Example 1.2. Let us consider two processes or phenomena of economic, social a.s.o. nature, and let us assume that the evolution in time of each of them both is the resultant of the evolutions of a certain number of factors. Taking as $\mathcal{S}_1$ and $\mathcal{S}_2$ the two factors sets, and considering the phenomenon or process having as components the two specified phenomena or processes, we can take as elements of $\mathcal{S}$ the groups of factors with at least two factors, the first one of the first category, the second one of the second category. As concerns the numerical structure of system $\mathcal{S}$, taking into consideration the mode it was built up, the following theorem is immediate.

Theorem 1.1. If systems $\mathcal{S}_1$ and $\mathcal{S}_2$, considered as blocks sets, are limited or at least numerable, then denoting by ν_{mn} the number of elements of type S_{mn} for m and n fixed, and by K the number of elements of the same type. contained by $\mathcal{S}$, following relations are valid:

$$\nu_{mn} = C_I^m \times C_J^n \text{ and } K = \sum_{m \in I} \sum_{n \in J} \nu_{mn} = (2^I - 1) \times (2^I - 1) \tag{1.2}$$

We notice that actually, $\mathcal{S}_{mn}$ is in its turn a system composed of ν_{mn} elements of the same form, thus containing all elements of m blocks each from $\mathcal{S}_1$, and n blocks each from $\mathcal{S}_2$.

Relation (1.2.) gives the maximum number of elements that can be contained by system $\mathcal{S}$, from the theoretical point of view, because as we saw in example 1.1., not all of them present interest in certain situations.

We shall show in what follows that regardless of their nature, the considered systems are not merely arbitrary sets of elements, but they can be endowed, as the case stands with certain properties.

Let us take as example system $\mathfrak{S}_1$. If the interpretation of its elements is the one in example 1.1., then we can define an order relation, by considering the eutrophy attached to each characteristic.

Definition 1.6. We shall say that, between the characteristics B_1^i and B_2^j, there is the relation "$\leqslant$" if $H(B_1^i) \leqslant H(B_2^j)$.

It is obvious that relation "$\leqslant$" is an order relation on $\mathfrak{S}_1$ and as a consequence $\mathfrak{S}_1$ is an ordered set.

Denoting by

$$B_1^i \wedge B_1^j = inf\{B_1^i, B_1^j\} \text{ and } \bigwedge_{i \in I} B_1^i = inf\{B_1^i\}_{i \in I} \tag{1.3.}$$

and by

$$B_1^i \vee B_2^j = sup\{B_1^i, B_1^j \text{ and } \bigvee_{i \in I} B_1^i = sup\{B_1^i\}_{i \in I} \tag{1.4.}$$

and taking into account the introduced order relation, it results that the elements defined by relations (1.3) and (1.4) do exist.

Consequently takes place:

Theorem 1.2. System $\mathfrak{S}_1$, with the considered interpretation is a lattice (i.e. a reticulated set).

2. TOPOLOGICAL CONSIDERATIONS

Let us now assume that system $\mathfrak{S}_1$ is arbitrarily chosen, and let us denote by $\mathfrak{Z}(\mathfrak{S}_1)$ the set of its parts, i.e. the set of all subsystems that can be composed from the blocks of $\mathfrak{S}_1$, thus containing both system $\mathfrak{S}_1$ and the void subsystem, with no block, denoted by Φ.

Definition 2.1. Let $\mathfrak{S}_1^{(1)}$ and $\mathfrak{S}_1^{(2)}$ be two arbitrary subsystems of $\mathfrak{S}_1$. We shall define their intersection as being the subsystem:

$$\mathfrak{S}_1^{(1)} \cap \mathfrak{S}_1^{(2)} = \{B_1^i / B_1^i \in \mathfrak{S}_1^{(1)} \text{ and } B_1^i \in \mathfrak{S}_1^{(2)}\} \subseteq \mathfrak{S}_1 \tag{2.1.}$$

and their union as the subsystem:

$$\mathfrak{S}_1^{(2)} \cup \mathfrak{S}_1^{(2)} = \{B_1^i / B_1^i \in S_1^{(1)} \text{ or } B_1^i \in \mathfrak{S}_1^{(2)}\} \subseteq \mathfrak{S}_1 \tag{2.2.}$$

Analogically we shall define the intersection and union for more than two subsystems or $\mathfrak{S}_1$, i.e. elements of $\mathfrak{Z}(\mathfrak{S}_1)$.

Theorem 2.1. System $\mathfrak{S}_1 = \{B_1^i\}_{i \in I}$ is a topological space.

Proof. As a matter of fact $\mathfrak{S}_1$ and the set of its parts $\mathfrak{Z}(\mathfrak{S}_1)$, being given, there is a topology $\tau_{\mathfrak{S}1}$ defined on $\mathfrak{S}_1$, because:

$$\text{i) } \Phi, \mathfrak{S}_1 \in \mathfrak{Z}(\mathfrak{S}_1)$$

ii) if $\{E_h\}_{h \in H}$ is a family whatever (non-void) of elements of $\mathfrak{Z}(\mathfrak{S}_1)$, and $E = \bigcup_{h \in H} E_h$, then $E \in \mathfrak{Z}(\mathfrak{S}_1)$, being according to the definition of the intersection a subsystem of $\mathfrak{S}_1$ hence an element of $\mathfrak{Z}(\mathfrak{S}_1)$.

iii) if $\{E_h\}_{h=1}^n$ with $E_h \in \mathfrak{Q}(\mathfrak{S}_1)$ and $E = \bigcap_{h=1}^{n} E_h$, then $E \in \mathfrak{Q}(\mathfrak{S}_1)$ being according to the definition of the intersection a subsystem of $\mathfrak{S}_1$, hence an element of $\mathfrak{Q}(\mathfrak{S}_1)$.

Definition 2.2. Any subsystem of $\mathfrak{S}_1$, as element of $\mathfrak{Q}(\mathfrak{S}_1)$ is called an open set, and one of its blocks is called point of topological space $\mathfrak{S}_1$ with topology $\tau_{\mathfrak{S}1}$.

Definition 2.3. A subsystem $F \subset \mathfrak{S}_1$ is called closed set, if set $E = CF$ is open.

Theorem 2.2. Let $\mathfrak{S}_1^{(0)} \subset \mathfrak{S}_1$ be a subsystem of $\mathfrak{S}_1$. Then the set

$$\mathfrak{Q}(\mathfrak{S}_1^{(0)}) = \{E \cap \mathfrak{S}_1^{(0)} / E \in \mathfrak{Q}(\mathfrak{S}_1)\} \tag{2.3.}$$

defines a topology $\tau_{\mathfrak{S}1}^{(0)}$ on $\mathfrak{S}_1^{(0)}$.

Proof. Indeed:

i) $\Phi, \mathfrak{S}_1^{(0)} \in \mathfrak{Q}(\mathfrak{S}_1^{(0)})$

ii) let be $\{G_h\}_{h \in H}$, $G_h \in \mathfrak{Q}(\mathfrak{S}_1^{(0)})$, $G_h \neq \Phi$, and $G = \bigcup_{h \in H} G_h$ Then, the subsystem $G = \bigcup G_h = \bigcup_{h \in H} (E_h \cap \mathfrak{S}_1^{(0)}) = (\bigcup_{h \in H} E_h) \cap \mathfrak{S}_1^{(0)} \in \mathfrak{Q}(\mathfrak{S}_1^{(0)})$ as $\bigcup_{h \in H} E_h \in \mathfrak{Q}(\mathfrak{S}_1)$, E_h, being also from $\mathfrak{Q}(\mathfrak{S}_1)$.

iii) let be $\{G_h\}_{h=1}^n$ $G_h \in \mathfrak{Q}(\mathfrak{S}_1^{(0)})$, $G_h \neq \Phi$ and $G = \bigcap_{h=1}^{n} G_h$. Then the subsystem

$$G = \bigcap_{h=1}^{n} G_h = \bigcap_{h=1}^{n} (E_h \cap \mathfrak{S}_1^{(0)}) = (\bigcap_{h=1}^{n} E_h) \cap \mathfrak{S}_1^{(0)} \in \mathfrak{Q}(\mathfrak{S}_1^{(0)}), \text{ as } \bigcap_{h=1}^{n} E_h \in \mathfrak{Q}(\mathfrak{S}_1).$$

Definition 2.4. Topology $\tau_{\mathfrak{S}1}^{(0)}$ is called a topology induced on $\mathfrak{S}_1^{(0)}$ by $\tau_{\mathfrak{S}1}$, and $\mathfrak{S}_1^{(0)}$ is called a topological subspace of $\mathfrak{S}_1$.

Definition 2.5. The topological space $\mathfrak{S}_1$ being given, a subsystem $V \subset \mathfrak{S}_1$ is called vicinity of the arbitrary block B_1^i, if there is a subsystem E such that $B_1^i \in E$ and $E \subset V$.

We shall denote by $V_{B_1^i}$ the set of all vecinities of B_1^i.

Theorem 2.3. As a topological space with topology $\tau_{\mathfrak{S}1}$, system $\mathfrak{S}_1$ is a separate or Hausdorff space.

Proof. It is obvious that, for any two blocks $B_1^i \neq B_1^j$, there is $V_{B_1^i}$ and $V_{B_1^j}$ such that $V_{B_1^i} \cap V_{B_1^j} = \Phi$, that is to say there are at least two subsystems of $\mathfrak{S}_1$, one containing the block B_1^i, the other containing the block B_1^j, such that their intersection be the void system.

Definition 2.6. We designate as covering of $\mathfrak{S}_1$ a family $\{E_h\}_{h \in H}$ of subsystems of $\mathfrak{S}_1$, having the property that $\mathfrak{S}_1 = \bigcup_{h \in H} E_h$. If the family contains a finished number of subsystem, we have finished covering of $\mathfrak{S}_1$.

Definition 2.7. A topological space is said to satisfy the Borel-Lebesgue condition, if from any of its coverings with open sets, one of its finished coverings can be extracted.

Let $\{E_h\}_{h\in H}$, $E_h \subset \mathfrak{S}_1$, be a covering of $\mathfrak{S}_1$ with open sets, i.e. $\mathfrak{S}_1 = \bigcup_{h\in H} E_h$. We denote by $\{G_l\}_{l=1}^n$ a family of subsystems of $\mathfrak{S}_1$ defined by relation

$$G_l = \bigcup_{h\in H_l} E_h, \quad H = \bigcup_{l=1}^{n} H_l \tag{2.4.}$$

It is obvious that $\mathfrak{S}_1 = \bigcup_{l\in 1}^{n} G_l$ and consequently the family $\{G_l\}_{l=1}^n$ is a finished covering of $\mathfrak{S}_1$.

If we define the family $\{G_l\}_{l=1}^n$ as being

$$G_l = \mathfrak{S}_1 - \bigcup_{l\in H_l} E_l, \quad \bigcup_{l=1}^{n} H_l = H \tag{2.5.}$$

we notice that $\bigcup_{l=1}^{n} G_l = \mathfrak{S}_1$, and we have also a finished covering of $\mathfrak{S}_1$.

Definition 2.8. A topological space is compact if it is separated and if it satisfies the Borel-Lebesgue condition.

In these conditions, the following theorem is obvious.

Theorem 2.4. The real system $\mathfrak{S}_1$ is a compact topological space.

According to theorem 2.2., any subsystem of $\mathfrak{S}_1$ is also topological space. Proceeding by analogy with relations (2.4.) or (2.5.), we can evince that any topological subspace of $\mathfrak{S}_1$ is also compact.

Definition 2.9. A subsystem $E \subset \mathfrak{S}_1$ is compact if the topological subspace E is compact.

It is analogically demonstrated that the second system $\mathfrak{S}_2$ is also at opological space with topology $\tau_{\mathfrak{S}_2}$.

Definition 2.10. The topological spaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are said to be homeomorphous if there is a biunivoc correspondence φ between $\mathfrak{S}_1$ and $\mathfrak{S}_2$ which preserve the sets open, i.e. $\varphi(\mathfrak{L}(\mathfrak{S}_1)) = \mathfrak{L}(\mathfrak{S}_2)$. Correspondence φ is called homeomorphism.

Let us now denote by

$$\mathfrak{S}_1^m = \mathfrak{S}_1 \times \mathfrak{S}_1 \times \ldots \times \mathfrak{S}_1 \tag{2.6.}$$

the space product of $\mathfrak{S}_1$ by itself m times, being known that a constitutive element of $\mathfrak{S}_1^m$ is composed of m blocks of $\mathfrak{S}_1$ and analogically as form and interpretation with

$$\mathfrak{S}_2^n = \mathfrak{S}_2 \times \mathfrak{S}_2 \times \ldots \times \mathfrak{S}_2 \tag{2.7.}$$

the space product of $\mathfrak{S}_2$ by itself n times.

Consequently, it results from relations (1.1.), (2.6.) and (2.7.) that

$$S_{mn} \subset \mathfrak{S}_1^m \times \mathfrak{S}_2^n, \quad m\in I, \quad n\in J \tag{2.8.}$$

and accordingly

$$\mathfrak{S} \subset \{(\mathfrak{S}_1^m \times \mathfrak{S}_2^n)/m\in I, \quad n\in J\} \tag{2.9.}$$

Denoting by $\mathfrak{S}_1^I \times \mathfrak{S}_2^J$ the space product of both spaces, product of types (2.6.) and (2.7.), it is obvious that

$$S_{IJ} \subset \mathfrak{S}_1^I \times \mathfrak{S}_J^2 \tag{2.10.}$$

Let us now do such that all elements of $\mathbb{S}$ be of dimension $I + J$, by completing with void blocks the lacking elements. Thus, an element S_{mn}, will be replaced by

$$S^*_{mn} = (S_{mn},\ \Phi_{I-m},\ \Phi_{J-n}) \tag{2.11.}$$

where the notation Φ_{I-m} means that we shall put element Φ $I - m$ times corresponding to the blocks of $\mathbb{S}_1$ and analogically for Φ_{J-n}.

In these conditions we shall have the system

$$\mathbb{S}^* = \{S_{mn}/m \in I,\ n \in J\} \tag{2.12.}$$

and, taking into account relation (2.9.), it results that

$$\mathbb{S}^* \subset \mathcal{P}(\mathbb{S}_1 \times \mathbb{S}_2) \tag{2.13.}$$

Tikhonov's theorem says that, having a compact spaces family, the produced topological space is also compact. Consequently, topological space $\mathbb{S}_1^I \times \mathbb{S}_2^J$ is compact, and $\mathbb{S}$ appears as a family of parts of the latter, i.e. of compact subspaces. Moreover, set $\mathbb{S}^*$ defines a topology on space $\mathbb{S}_1^I \times \mathbb{S}_2^J$.

3. PROBABILISTIC CONSIDERATIONS

Let us assume that system $\mathbb{S}_1$ is evolutive, meaning by this that it undergoes modifications in time, as concerns the values its components can take in certain moments. We are not interested neither in the evolution equations, nor in the conditions in which this evolution takes place.

If, at a given moment, a block, $B_1^i \in \mathbb{S}_1$, can take certain values, not necessarily numbers, let us denote by Ω_i their set, i.e.

$$\Omega_i = \{\omega_i^1, \omega_i^2, ..., \omega_i^{h_i}, ...\} = \{\omega_i^{h_i}\}_{h_i \in H_i} \quad i \in I. \tag{3.1.}$$

Definition 3.1. The situation of an evolutive system at a given moment determined by the set of the instantaneous values its blocks can take, is called system's state.

The state of system $\mathbb{S}_1$, at a given moment, can be hence represented by a point with I coordinates.

It is obvious that, if the system's evolution is deterministic, and the evolution law is known, then the point representing the system's state at a given moment is known and unique. But if the deterministic law is not known, or if the evolution is stochastic, as the point representing the system's state at a given moment is not known by us and cannot be determined with certainty, it can be considered variable and consequently there is a set of possible states.

Thus, the set of system's, $\mathbb{S}_1$ is a set Ω given by relation

$$\Omega = \prod_{i \in I} \Omega_i \tag{3.2.}$$

We consider now the set Ω_i and let $\mathcal{K}_i \subset \mathcal{P}(\Omega_i)$ be a σ — body of parts. Let also P_i be a probability defined on $\mathcal{K}_i$, according to Kolmogorov's axioms. We have hence a field of probability $\{\Omega_i, \mathcal{K}_i, P_i\}$, and consequently we can

consider the family of fields of probability $\{\Omega_i, \mathfrak{K}_i, P_i\}_{i\in I}$. Let now $\{\Omega^I, \mathfrak{K}^I, P^I\}$ be the produced probability field, whose elements are defined as follows:

Set Ω^I is the cartesian product of sets Ω_i given by relation (3.1.).

Definition 3.2. Any part of Ω^I whose form is

$$\prod_{i\in I} A_i \times \prod_{i\in I\setminus I_1} \Omega_i \tag{3.3.}$$

where $A_i \in \mathfrak{K}_i$, $i \in I_1 \subset I$, is called cylinder.

Denoting by $\mathfrak{F}^I$ the set of all cylinders from Ω^I, and by $\mathcal{G}^I$ the set of unions finished by the cylinders from $\mathfrak{F}^I$, one can evince that $\mathcal{G}^I$ is a body of parts. We'll take as $\mathfrak{K}^I$ the smallest borelian body containing $\mathcal{G}^I$, hence generated by $\mathcal{G}^I$.

We define probability P^I as follows

$$P^I(\prod_{i\in I_1} A_i \times \prod_{i\in I\setminus I_1} \Omega_i) = \prod_{i\in I_1} P_i(A_i) \cdot \prod_{i\in I\setminus I_1} P_i(\Omega_i) = \prod_{i\in I_1} P_i(A_i) \tag{3.4.}$$

We hence defined a probability and consequently, for a state whatever ω^0 of the system, we have relation

$$P^I(\mathfrak{s}_1 = \omega^0) = P^I(B_1^1 = \omega_1^0,\ B_1^2 = \omega_2^0, \ldots) \tag{3.5.}$$

Definition 3.3. We call random variable, defined by a borelian events field $\{\Omega_i, \mathfrak{K}_i\}$, an application $X_i : \Omega_i \to R$ with the property that $X_i^{-1}(A_i) \in \mathfrak{K}_i$ for any $A_i \in \mathfrak{R}$, where $\mathfrak{R}$ is the borelian body of the real sets.

We shall denote by x_i the values of application X_i, i.e.:

$$X_i(\omega_i^{h_i}) = x_i^{h_i},\ h_i \in H_i \tag{3.6.}$$

and consequently, for a fixed h, we have

$$P_i(X_i = x_i^h) = P_i(B_1^i = \omega_i^h) \tag{3.7.}$$

Analogically with definition 3.3. we also define a random multidimensional variable, or a random vector. We shall denote by X the random vector whose components are the random variables X_i, $i \in I$. For a fixed h we can write the relation

$$P^I(X = x^h) = P^I(X_1 = x_1^h,\ X_2 = x_2^h, \ldots, X_i = x_i^h, \ldots) \tag{3.8.}$$

and denoting by

$$p_i^{h_i} = P_i(X_i = x_i^{h_i}),\ h_i \in H_i,\ i \in I \tag{3.9.}$$

then

$$\sum_{h_i \in H_i} p_i^{h_i} = 1,\ i \in I \tag{3.10.}$$

Analogically with the above stated, we can set the problem also for system $\mathfrak{s}_2$. We shall denote by

$$\Gamma_j = \{\gamma_j^1, \gamma_j^2, \ldots, \gamma_j^{l_j} \ldots\} = \{\gamma_j^{l_j}\}_{l_j \in L_j} \tag{3.11.}$$

the set of values a block whatever B_2^j of system $\mathfrak{s}_2$ can take.

We can consider the probability field $\{\Gamma_j, \overline{\mathfrak{K}}_j, Q_j\}$ as well as the produced field $\{\Gamma^J, \overline{\mathfrak{K}}^J, Q^J\}$, whose elements are defined as in the case of the first system. We shall denote by Y_j a random variable defined on field $\{\Gamma_j, \overline{\mathfrak{K}}_j\}$ and by Y the random vector defined on $\{\Gamma^J, \overline{\mathfrak{K}}^J\}$, whose components are the random variables Y_j, $j \in J$.

Denoting by $y_j^{l_j}$ the values of variable Y_j and by $q_j^{l_j}$ their probabilities, we have

$$q_j^{l_j} = Q_j(Y_j = y_j^{l_j}); \quad \sum_{l_j \in L_j} q_j^{l_j} = 1 \tag{3.12.}$$

and for any l:

$$Q^J(Y = y^l) = Q^J(Y_1 = y_1^{l_1},\ Y_2 = y_2^{l_2}, \ldots, Y_j = y_j^{l_j}, \ldots) \tag{3.13.}$$

We can analogically proceed for system $\mathfrak{S}$ too. Thus, denoting by

$$\Omega_i \times \Gamma_j = \{(\omega_i^{h_i}, \gamma_j^{l_j}) / \omega_i^{h_i} \in \Omega_i, \gamma_j^{l_j} \in \Gamma_j\}_{h_i \in H_i,\ l_j \in L_j} \tag{3.14.}$$

the set of all values the blocks pair $(B_1^i,\ B_2^j) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ can take, we can build up a produced probabilities field, and let Π_{ij} be a probability defined on this field, i.e.

$$\Pi_{ij}[(X_i,\ Y_j) = (x_i^{h_i}, y_j^{l_j})] = \Pi_{ij}[X_i = x_i^{hi},\ Y_j = y_j^{lj}] \tag{3.15.}$$

and denoting by $\pi_{ij}^{h_i l_j}$ this probability, it results that

$$\sum_{\substack{h_i \in H_i \\ l_j \in L_j}} \pi_{ij}^{h_i l_j} = 1; \quad \sum_{h_i \in H_i} \pi_{ij}^{h_i l_j} = q_j^{l_j}; \quad \sum_{l_j \in L_j} \pi_{ij}^{h_i l_j} = p_i^{h_i} \tag{3.16.}$$

As an immediate generalization of relation (3.14) we'll have relation

$$\prod_{i \in I} \Omega_i \times \prod_{j \in J} \Gamma_j \tag{3.17.}$$

that represents the set of values the elements of $\mathfrak{S}$ can take.

Considering the random vector (X, Y), we shall denote by Π^{IJ} the probability defined on the produced field built over set (3.17.) and consequently, for fixed h and l,

$$\Pi^{IJ}[(X, Y) = (x^h, y^l)] = \Pi^{IJ}[(X_1 = x_1^h,\ X_2 = x_2^h \ldots),\ (Y_1 = y_1^l,\ Y_2 = y_2^l \ldots)] \tag{3.18.}$$

With a corresponding notation as for relation (3.15.) we shall find the generalization of relations (3.16).

In the following, without making any confusion, we shall adopt a single notation, namely

$$p(x) = P(X = x),\ q(y) = P(Y = y) \text{ and } \pi(x, y) = P(X = x, Y = y) \tag{3.19.}$$

while we understand every time that this is a matter of the probabilities written in detail above.

4. CORRESPONDENCE BETWEEN A REAL SYSTEM AND ITS PROBABILISTIC MODEL

We come now back to systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, and we denote by $\mathcal{A}$ the set of all random variables, and by $\mathcal{P}(\mathcal{A})$ the set of its parts, whose elements are random vectors.

Denoting by w the elements of $\mathcal{A}$ and by W those of $\mathcal{P}(\mathcal{A})$, then for two arbitrary elements we shall define the union as being

$$W_1 \cup W_2 = \{w \in \mathcal{A} / w \in W_1, \text{ or } w \in W_2\} \in \mathcal{P}(\mathcal{A}) \tag{4.1.}$$

and their intersection as

$$W_1 \cap W_2 = \{w \in \mathcal{A} / w \in W_1 \text{ and } w \in W_2\} \in \mathcal{P}(\mathcal{A}) \tag{4.2.}$$

We can analogically define the union and intersection for more than two elements of $\mathcal{P}(\mathcal{A})$.

Theorem 4.1. The set of all random variables $\mathcal{A}$ is a topological space.

We shall denote by $\tau_{\mathcal{A}}$ the topology defined on $\mathcal{A}$. Any random vector W as element of $\mathcal{P}(\mathcal{A})$ is called open set.

Let now be $\mathcal{A}_1 \subset \mathcal{A}$.

Theorem 4.2. The set

$$\tau_{\mathcal{A}_1} = \{W \cap \mathcal{A}_1 / W \in \mathcal{P}(\mathcal{A})\} \tag{4.3.}$$

defines a topology on $\mathcal{A}_1$.

The topology defined in theorem 4.2. is called the topology induced on $\mathcal{A}_1$ by the topology of $\mathcal{A}$, and $\mathcal{A}_1$ makes up a topological subspace of $\mathcal{A}$.

Let us consider as element of set $\mathcal{A}_1$, the random variables X introduced by definition 3.2.

Let $\varphi_1 : \mathcal{S}_1 \to \mathcal{A}_1$, be a function, that makes that to every block $B_1^i \in \mathcal{S}_1$ corresponds a single random variable, and reciprocally.

As the correspondence $\mathcal{S}_1$ makes that to every block corresponds a random variable, it results that to every subsystem of $\mathcal{S}_1$ will correspond a vector of $\mathcal{P}(\mathcal{A}_1)$, that is, to any open set from $\tau_{\mathcal{S}_1}$ corresponds through φ_1, an open set from $\tau_{\mathcal{A}_1}$. As $\varphi_1(\mathcal{S}_1) = \mathcal{A}_1$, it results that $\varphi_1(\tau_{\mathcal{S}_1}) = \tau_{\mathcal{A}_1}$.

Theorem 4.3. The system $\mathcal{S}_1$ and the set of the random variables corresponding to its components, $\mathcal{A}_1$, are two homeomorphous topological spaces.

As can be seen, theorem 4.3. ensures the correspondence between the real system $\mathcal{S}_1$ and the set of random variables $\mathcal{A}_1$, with whose help the system $\mathcal{S}_1$ is actually studied. Considered also as a system, the set $\mathcal{A}_1$ makes up an image system of $\mathcal{S}_1$, through transformation φ_1. Carrying out observations on $\mathcal{S}_1$, we process them through the agency of $\mathcal{A}_1$, and the obtained results form information about $\mathcal{S}_1$. Any modification in $\mathcal{S}_1$ brings about modifications in the observations, hence in the data we process in $\mathcal{A}_1$, and consequently modifications in the information about $\mathcal{S}_1$.

With these considerations, we can say that $\mathcal{A}_1$, as a system, represents the probabilistic model of system $\mathcal{S}_1$, and φ_1 makes up the correspondence between system $\mathcal{S}_1$ and its model $\mathcal{A}_1$.

Hence, any modifications effected on the mathematical model implies the appearance or the loss of some information concerning the system $\mathcal{S}_1$.

Theorem 4.3. and the subsequent considerations can be analogically presented also for systems $\mathcal{S}_2$ and $\mathcal{S}$.

5. ESTIMATION OF A SYSTEM STATE

Let f: $R^I \times R^J \to R$ be an integrable Lebesgue function.

Definition 5.1. We shall say that function f is distribution density for the random vector (X, Y) if

$$f(x, y) \geq 0, \quad \int_{R^I}\int_{R^J} f(x, y)\mathrm{d}x\,\mathrm{d}y \tag{5.1.}$$

$$\text{where } \mathrm{d}x = \prod_{i \in I}\mathrm{d}x_i, \quad \mathrm{d}y = \prod_{j \in J}\mathrm{d}y_i.$$

The distributions of the random vectors X and Y, as marginal distributions, will be given by the relations

$$g(x) = \int_{R^J} f(x, y)\mathrm{d}y \text{ and } h(y) = \int_{R^I} f(x, y)\mathrm{d}x \tag{5.2.}$$

Speaking about the X — random vector distribution, we actually think at the distribution of the point representing the state of system $\mathcal{S}_1$, at the moment when carrying out the study, and analogically for Y or (X, Y).

Considering the system S_1, the respective point with I coordinates belongs to the corresponding I — dimensional space, called selection space.

But taking the random vector X attached to $\mathcal{S}_1$, the point belongs to the I — dimensional euclidian space R^I, that contains or even can be itself selection space.

Denoting by $\mathfrak{X}_i$ the selection space corresponding to the random variable X_i, the selection space corresponding to the random vector X is given by relation

$$\mathfrak{X} = \prod_{i \in I}\mathfrak{X}_i \subseteq R^I \tag{5.3.}$$

Definition 5.2. The block $B_1^i \in \mathcal{S}_1$ being given, an arbitrary element $\omega_i^{h_i} \in \Omega_i$ is called alternative of the considered block, and set Ω_i forms the set of the alternatives wherein block B_1^i can be present at the considered moment.

Definition 5.3. The set of all alternatives wherein the considered block can be found is called selection space corresponding to the block.

Definition 5.4. The choice of one or several alternatives bears the name of selection.

Analogically is defined the selection also for the random variable attached to the considered block.

If n_i alternatives are selected, we shall have a volume selection n_i and we shall consider as selection space the set $\Omega_i^{n_i}$, given by the relation

$$\Omega_i^{n_i} = \underbrace{\Omega_i \times \Omega_i \times \ldots \times \Omega_i}_{n_i} \tag{5.4.}$$

and corresponding to values of variable X_i, the selection space will be in this case

$$\mathfrak{X}_i^{n_i} = \underbrace{\mathfrak{X}_i \times \mathfrak{X}_i \times \ldots \times \mathfrak{X}_i}_{n_i} \tag{5.5.}$$

In these conditions the selection space corresponding to system $\mathcal{S}_1$ will be

$$\Omega = \prod_{i \in I} \Omega_i^{ni} \tag{5.6.}$$

and analogically for the random vector X we shall have

$$\overline{\mathfrak{X}} = \prod_{i \in I} \mathfrak{X}_i^{n_i} \subset \mathrm{R}^n, n = \sum_{i \in I}^{n_i} \tag{5.7.}$$

Let now $T: \overline{\mathfrak{X}} \to \mathrm{R}^{I_1}$ be a Borel measurable function.

Definition 5.5. Function T is called selection function or statistics.

During its evolution in time, the system can have at a given moment, a certain state, i.e. every block could be found in a certain, precise alternative, but which we don't know and consequently we suppose that it is one of set of considered alternatives. We shall call this state theoretical state and any other state we assume as being theoretical, will be called approximate system's state or estimation of the theoretical state of the system.

As the information on the system is obtained through the agency of the random vector X, the information providing observations belong to $\overline{\mathfrak{X}}$, and practically to obtain the estimation of a system state means to effect a selection from $\overline{\mathfrak{X}}$ through the agency of an application defined on the selection set, and having values in the set of states.

Definition 5.6. The statistics, with the help of which an estimation is carried out, will be called an estimation function or estimator.

Let us now presume that, at the given moment, the considered systems pair, $\mathcal{S}_1$ and $\mathcal{S}_2$, satisfies the following axioms:

(A.1) system $\mathcal{S}_1$ can have various states, we can know and study as a result of some observations;

(A.2) system $\mathcal{S}_2$ has a single state, precisely determined, but unknown to us, the state point being in $\mathbf{R}^I$;

(A.3) the probabilistic distribution of the states of system $\mathcal{S}_1$ depends on the state of system $\mathcal{S}_2$, understanding by this that, for a certain state of system $\mathcal{S}_2$, the distribution density corresponding to system $\mathcal{S}_1$ has a certain expression;

(A.4) the state of system $\mathcal{S}_2$ can be estimated with the help of the states of system $\mathcal{S}_1$, that is to say there exist the statistics T: $\overline{\mathfrak{X}} \to \mathbf{R}^{I_1}$, through the agency of which, to one selection from Ω, actually performed from $\mathfrak{X}$, a state of system $\mathcal{S}_2$ is corresponding.

As, according to postulate (A.2), the state of system $\mathfrak{S}_2$ is not known, and by trying to estimate by means of the states of $\mathfrak{S}_1$, conformably to the fourth postulate, it can appear in various situations, we shall assume it to be further stochastic, and let us denote by $Y \subset \mathbf{R}^I$ the set of these situations.

Definition 5.7. We shall say that the probabilistic distribution $f(x, y)$ is specified if the state of system $\mathfrak{S}_2$ is not known, and contrarily we'll say it is completely specified.

Various methods and several types of estimations are available for estimating the state of system $\mathfrak{S}_2$. Further on, we shall deal only with a certain type of estimators, taking into account a property assigned only to them.

Let us assume that we carry out a volume selection k for any of the random variables X_i, $j \in I$, hence $n_i = k$ and consequently the selection space will be now

$$\overline{\mathfrak{X}} = \prod_{i \in I} \mathfrak{X}_i^k \tag{5.8.}$$

For one on the observations, the distribution density value is $f(x^i, y)$, and let us denote by

$$P(x, y) = \prod_{i=1}^{k} f(x^i, y) = P(x^1, x^2, ..., x^k, y) \tag{5.9.}$$

the likelihood function, that means the probability of simultaneously achieving the k independent tests carried out on $\mathfrak{S}_1$ and where results are $x^1, x^2, ..., ..., x^k$.

As in the expression of the likelihood function there appears at least one function of the k observations, $T(x) = T(x^1, x^2, ..., x^k)$, whose form does not matter now, we can write relation (5.9) also like this

$$P(x, T(x), y) = P(x^1, x^2, ..., x^k, T(x^1, x^2, ..., x^k), y) = \prod_{i=1}^{k} f(x^i, y) \tag{5.10.}$$

Let us denote by

$$P(x/(T, y)) = \frac{P(x, T, y)}{P(T, y)} \tag{5.11.}$$

the probability of achieving the k tests, conditioned by the values $T(x)$ and y.

In the conditions in which $T: \overline{\mathfrak{X}} \to \mathrm{R}^{I_1}$, let us denote the image of $\overline{\mathfrak{X}}$ through T by $\mathfrak{T}$, i.e. $\mathfrak{T} = T(\mathfrak{X})$, and we shall admit that the unique state of $\mathfrak{S}_2$, unknown to us, can be estimated by any of the elements of $\mathfrak{T}$.

Definition 5.8 .We shall say that the statistics $T(x)$ is sufficient for the distribution family $\{f(x, y)\}_{y \in Y}$, or otherwise expressed, that they make out a sufficient estimator for the state of system $\mathfrak{S}_2$, if the probability $P(x/(T, y))$ does not depend on y, that is

$$P(x/(T, y)) = \psi(x) = \psi(x^1, x^2, ..., x^k) \tag{5.12.}$$

Taking into account the relations (5.11.) and (5.12), the sufficiency relation may be also written under the form

$$P(x, T, y) = P(T, y)\psi(x) \tag{5.13.}$$

We shall present further on an important property of the sufficient estimators.

6. INFORMATIONAL CONSIDERATIONS

Let then $f(x, y)$ be the distribution density of the states of system $\mathcal{S}_1$ at a given moment, whose expression depends on the values of y, that represent the state of system $\mathcal{S}_2$, and $g(x)$ and $h(y)$ are the corresponding marginal distribution densities.

Let us consider now the fields $\mathcal{D} \subset \mathrm{R}^I$ and $\mathcal{E} \subset \mathrm{R}^J$ and suppose that $f(x, y)$ is positive and continuous on $\mathcal{D} \times \mathcal{E}$.

Definition 6.1. If it exists, the integral

$$I(x, y) = \iint_{(\mathcal{D},\mathcal{E})} f(x, y) \log \frac{f(x, y)}{g(x)h(y)} \, dx \, dy \tag{6.1.}$$

represents the amount of information contained by the states of system $\mathcal{S}_1$, concerning the state of system $\mathcal{S}_2$ or reversely.

Let now $T: \overline{\mathcal{X}} \to \mathrm{R}^{I_1}$, $I_1 \subset I$ be an estimator for the state of system $\mathcal{S}_2$, and $\mathcal{T} = T(\mathcal{X}) \subset \mathrm{R}^I$.

Considering T as a random variable, let $f_1(t, y)$ be the distribution density of random vector (T, Y), and $g_1(t)$ the marginal distribution density of T, f_1 being integrable on $\mathcal{T} \times \mathcal{E}$.

In these conditions we can write a relation similar to (6.1.), namely

$$I(t, y) = \iint_{(\mathcal{T},\mathcal{E})} f_1(t, y) \log \frac{f_1(t, y)}{g(t)h(y)} \, dt \, dy \tag{6.2.}$$

and if this integral exists, it will represent the information amount contained by estimator $T(x)$ with respect to state y it is estimating.

The most important connection between relations (6.1.) and (6.2.) is given by the following theorem.

Theorem 6.1. If estimator T is sufficient for the state of system $\mathcal{S}_2$, then expressions (6.1.) and (6.2.) are equals, and consequently the information amount contained by the effected observations relating to y is the same as that contained by the considered sufficient estimator.

Proof. Let us introduce on $\mathcal{D}$ a local coordinate system (t, u), where $t \in \mathcal{T} \subset \mathrm{R}^I$, and $u \in \mathcal{D} \setminus \mathcal{T}$. Let us assume the jacobian of the $x \to (t, u)$ transformation does exist, is continuous and non-zero for $x \in \mathcal{D}$. As a result, relation (6.1) can be written as follows:

$$I(x, y) = \iiint_{(\mathcal{T},\mathcal{D}\setminus\mathcal{T},\mathcal{E})} f(t, u, y) \log \frac{f(t, u, y)}{g(t, u)h(y)} \, dt \, du \, dy = I(t, u, y) \tag{6.3.}$$

As T is sufficient, we have

$$f(t, u, y) = a(t, y)b(t, u) \neq 0 \tag{6.4.}$$

and as a result

$$I(t, u, y) = \iiint_{(\mathcal{T}, \mathcal{D} \setminus \mathcal{T}, \mathcal{S})} a(t, y)b(t, u) \log \frac{a(t, u)b(t, u)}{g(t, u)h(y)} \, dt \, du \, dy \tag{6.5.}$$

But as

$$g(t, u) = \int_{\mathcal{S}} f(t, u, y) dy = b(t, u) \int_{\mathcal{S}} a(t, y) dy \tag{6.6.}$$

it results that

$$I(t, u, y) = \iiint_{(\mathcal{T}, \mathcal{D} \setminus \mathcal{T}, \mathcal{S})} a(t, y)b(t, u) \log \frac{a(t, u)}{h(y) \int_{\mathcal{S}} a(t, y) dy} \, dt \, dy \tag{6.7.}$$

and by integrating with respect to u, we deduce that

$$I(t, u, y) = \iint_{(\mathcal{T}, \mathcal{S})} a(t, y)b_1(t) \log \frac{a(t, y)}{h(y) \int_{\mathcal{S}} a(t, y) \, dy} \, dt \, dy \tag{6.8.}$$

where

$$b_1(t) = \int_{\mathcal{D} \setminus \mathcal{T}} b(t, u) du \tag{6.9.}$$

Let us now denote by

$$f_1(t, y) = a(t, y)b_1(t) \tag{6.10}$$

and let us notice that this is the density probability for the random vector (T, Y).

Denoting by

$$g_1(t) = \int_{\mathcal{S}} f_1(t, y) dy \tag{6.11.}$$

the marginal distribution density of T, and taking into account relation (6.10), relation (6.8.) becomes

$$I(t, y) = \iint_{(\mathcal{T}, \mathcal{S})} f_1(t, y) \log \frac{f_1(t, y)}{h(y)g_1(t)} \, dt \, dy \tag{6.12.}$$

that is to say just relation (6.2), and the theorem is demonstrated.

Let us now denote by $H(X)$, $H(Y)$ and $H(X, Y)$ the entropies corresponding to the considered random vectors. With these notations relation (6.1.) can be also written in the form

$$I(x, y) = H(X) + H(Y) - H(X, Y) = J(X, Y) \tag{6.13.}$$

and the expression $J(X, Y)$ is called interdependence between the random vectors X and Y or, more generally, between systems $\mathcal{S}_1$ and $\mathcal{S}_2$, what, through an abuse of notation, could be written as follows

$$J(\mathcal{S}_1, \mathcal{S}_2) = H(\mathcal{S}_1) + H(\mathcal{S}_2) - H(\mathcal{S}_1, \mathcal{S}_2) \tag{6.14.}$$

Analogically, relation (6.2.) can be written in the form

$$J(T, Y) = H(T) + H(Y) - H(T, Y) \tag{6.15.}$$

where $T = T(x)$ is a function $T: \mathcal{A}^I \to \mathcal{A}^{I_1}$, $\mathcal{A}$ being the set of all random variables, and $I_1 \subset I$, and through the same abuse of notation we shall write

$$J(T(\mathcal{S}_1), \mathcal{S}_2) = H(T(\mathcal{S}_1)) + H(\mathcal{S}_2) - H(T(\mathcal{S}_1), \mathcal{S}_2) \tag{6.16.}$$

where $T(\mathcal{S}_1)$ means the image of $\mathcal{S}_1$ through transformation T, with the estimation interpretation of $\mathcal{S}_2$, generalizing here the estimation of state y of the latter.

If we interpret relation (6.15) as being the interdependence between estimation and the parameter to be estimated, then relation (6.16) has the significance of interdependence between the system to be estimated $\mathcal{S}_2$ and its estimation $T(\mathcal{S}_1)$. Taking into account the properties of entropy, the above relations can be written as follows:

$$J(X, Y) = H(X) - H(X/Y) = H(Y) - H(Y/X) \tag{6.17.}$$

$$J(\mathcal{S}_1, \mathcal{S}_2) = H(\mathcal{S}_1) - H(\mathcal{S}_1/\mathcal{S}_2) = H(\mathcal{S}_2) - H(\mathcal{S}_2/\mathcal{S}_1) \tag{6.18.}$$

$$J(T, Y) = H(T) - H(T/Y) = H(Y) - H(Y/T) \tag{6.19.}$$

$$J(T(\mathcal{S}_1), \mathcal{S}_2) = H(T(\mathcal{S}_1)) - H(T(\mathcal{S}_1)/\mathcal{S}_2) = H(\mathcal{S}_2) - H(\mathcal{S}_2/T(\mathcal{S}_1)) \tag{6.20.}$$

In these conditions, the following consequences of theorem (6.1.) are immediate.

Consequence 6.1.1. The non-determination of system $\mathcal{S}_2$ conditioned by $\mathcal{S}_1$ or by $T(\mathcal{S}_1)$ is the same, i.e.

$$H(\mathcal{S}_2/\mathcal{S}_1) = H(\mathcal{S}_2/T(\mathcal{S}_1)) \tag{6.21.}$$

In another interpretation, relation (6.21) says that the information contained by $\mathcal{S}_1$ and $T(\mathcal{S}_1)$ with respect to $\mathcal{S}_2$ is the same, of course when T is a sufficient estimator.

Consequence 6.1.2. The information amount variation when passing from system $\mathcal{S}_1$ to system $T(\mathcal{S}_1)$ is the same, no matter whether both systems are dependent or independent of system $\mathcal{S}_2$, i.e.

$$H(T(\mathcal{S}_1)) - H(\mathcal{S}_1) = H(T(\mathcal{S}_1)/\mathcal{S}_2) - H(\mathcal{S}_1/\mathcal{S}_2) \tag{6.22.}$$

Let us notice that all considerations from theorem 6.1. and from its consequences are made in the hypothesis of an existing interdependence between $\mathcal{S}_1$ and $\mathcal{S}_2$.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are independent, all above relations get trivial.

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QUELQUES CONSIDERATIONS DE LA THEORIE MATHEMATIQUE DES SYSTEMES

On présente quelques considérations topologiques, probabilistiques et informationnelles concernant des systèmes réels se trouvant en interdépendance, traitant aussi du problème de l'estimation de l'état d'un système.

НЕКОТОРЫЕ СООБРАЖЕНИЯ ПО МАТЕМАТИЧЕСКОЙ ТЕОРИИ СИСТЕМ

В этой статьи приведены некоторые топологические, вероятностные и информационные соображения по реальным взаимозависящим системам, ставя и задачу оценки состояния системы.

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