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Solving MER Optimal Control Problem via Mathematical Programming

In the present paper the authors complete the previous results concerning the efficiency of solving the minimum energy regulator (MER) problem via mathematical programming.

There are presented the results regarding the use of BEALE's algorithm in solving the quadratic programming model obtained by a first order discrete formulation of the model and of that obtained by a RAYLEIGH-RITZ aproximation of solution.

At the same time, the corresponding model with a view to apply Balakrishanan's technique and the Solution Optained by Solving a reduced quadratic programming problem is given.

The optimal minimum energy, function of the intercept as a parameter is studied also.

THE MINIMUM ENERGY REGULATOR (MER) PROBLEM

The model of the minimum energy regulator problem proposed in [10] and [5] is:

$$\text{minimize } J = \int_{t_0}^{t_f} u^2(t)\ \mathrm{d}t$$

such that:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

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with the boundary conditions

$$x_1(0) = -1 \qquad x_1(t_f) = 0$$
$$x_2(0) = 0 \qquad x_2(t_f) = 0$$
$$t_0 = 0 \qquad t_f = 2.20$$

plus the additional constraints:

$$g(x, t) = x_1 + 3x_2 + 0.02t - 1 \cdot 5 \leqslant 0$$
$$-1 \leqslant u \leqslant 1$$

FIRST ORDER DISCRETE FORMULATION OF THE MINIMUM ENERGY REGULATOR PROBLEM

Discretizing the problem with a first order approximation and by dividing the $[t_0, t_f]$ interval into $m = 30$ equal intervals of length Δt, Descheneau [5] reduces the problem to a quadratic programming model M, with 92 constraints and 30 variables:

$$min\ J = \Delta t \sum_{i=0}^{m-1} u^2(i)$$

suject to:

$$u(i) - 1 \leqslant 0 \qquad i = 0,\ 1,\ ...,\ m-1$$
$$-u(i) - 1 \leqslant 0 \qquad i = 0,\ 1,\ ...,\ m-1$$

$$\sum_{k=0}^{i-1} [(\Delta t)^2\ (i - k - 1) + 3\ \Delta t] \cdot u(k) + 0 \cdot 02(i \cdot \Delta t) - 2 \cdot 5 \leqslant 0 \quad i = 1\ ,...,\ m$$

$$-(\Delta t)^2 \sum_{k=0}^{m-2} (m - k - 1)\ u(k) + 1 = 0$$

$$-\Delta t \sum_{k=0}^{m-1} u(k) = 0$$

In fact in [5] Descheneau studied a model, which we will call M_0, with the last two equations taken with $\leqslant$ inequality signes, as we considered that $x_1(m) = 0$, $x_2(m) = 0$ boundary conditions are active one side only.

In [8] we are studying the full model, $M1$, which in the M_0 model with the last equations taken as:

$$-\varepsilon^2 \leqslant \Delta t^2 \sum_{k=0}^{m-2} (m - k - 1)\ u(k) - 1 \leqslant \varepsilon^2$$

$$-\varepsilon^2 \leqslant \Delta t \sum_{k=0}^{m-1} u(k) \leqslant \varepsilon^2$$

i.e. a model with 94 constraints and 30 variables.

If the solving technique needs the knowledge of an interior point, in order to find out in an easiest way one of it, then we have considered as in [5] a version M_{12} of model M_1, viz.:

$$min\ J = \Delta t \sum_{i=0}^{m-1} u^2(i) + M_{s^2}$$

subject to:

$$u(i) - 1 \leqslant 0 \qquad i = 0,\ 1,\ ...,\ m-1$$

$$-u(i) - 1 \leqslant 0 \qquad i = 0,\ 1,\ ...,\ m-1$$

$$\sum_{k=0}^{i-1} [(\Delta t)^2\,(i-k-1) + 3\,\Delta t]\,u(k) + 0 \cdot 02(i \cdot \Delta t) - 2 \cdot 5 \leqslant 0, \quad i = 1, ..., m$$

$$-s^2 \leqslant \Delta t^2 \sum_{k=0}^{m-2} (m-k-1)\,u(k) - 1 \leqslant s^2$$

$$-s^2 \leqslant \Delta t \sum_{k=0}^{m-1} u(k) \leqslant s^2$$

i.e. a model with 94 constraints and 31 variables.

Finally, we studied [8] the efficiency of solving the problem for model M_2, equivalent to M_1, where in the first two conditions, $u(i) - 1 \leqslant 0$, $-u(i) - 1 \leqslant 0$, $i = 0,\ 1,\ ...,\ m-1$, were replaced by $[u(i)]^2 - 1 \leqslant 0$, $i = 0,\ 1,\ ...,\ m-1$, i.e. a model with 64 constraints, among which 30 are quadratic, instead of linear ones, and 30 variables A model, called M_{22} equivalent to M_2 but analogoas to M_{12}, obtained by introducing the artificial variable s with a big enough penalty coefficient, have been also studied. The model M_{22} has 64 constraints and 31 variables.

The numerical tests showen [8] that the best results in solving the problem by SUMT technique are given by the M_2 model. These results, which in fact are still inaccurate regarding the boundary condition $x_1(m) = 0$, are graphed in figs. 1 and 2. The graphs regard the results obtained using the Davidon-Fletcher-Powell unconstrained algorithm in four sequences of SUMT techniques, with the Strodiot-Nguyen penalty [11].

Figures 3 and 4 show the result obtained, graphing the solution of the M model given in table 1, by the Beale's algorithm.

Comparing the both, we are infering the obvious conclusion, mentioned by us and other also in other occasions, that for a particularly structured problem a special designed algorithm gives better results. In order to use the Beale's algorithm we performed the following easy understanding transformations: $u = u' - 1$, $0 \leqslant u' \leqslant 2$. Making these transformations is worth to mention that the M model with Beale's algorithm is reducing only to 62 equations. The model has been solved on a IBM 370/135 by Land and Powell [12] Beale's QP programme in 3 min CPU total time.

Table 1

The solution of M Model for MER problem in 33 iterations, by Beale's Q. P. Algorithm

i	$u(i)$	$x_2(i+1) = x_2(i) + tu_2(i)$	$x_1(i+1) = x_1(i) + tx_2(i)$	i	$u(i)$	$x_2(i+1)$	$x_1(1+1)$
0		0	−1.0000	16	−0.23	0.6552	−0.4418
1	1.	0.0732	−1.0000	17	−0.53	0.6114	−0.3938
2	1.	0.1466	−0.9947	18	0.09	0.6179	−0.3490
3	1.	0.2199	−0.9840	19	−0.22	0.6018	−0.3038
4	1.	0.2932	−0.9679	20	−0.21	0.5865	−0.2597
5	1.	0.3665	−0.9465	21	−0.21	0.5712	−0.2168
6	1.	0.4398	−0.9197	22	−0.20	0.5566	−0.1750
7	1.	0.5131	−0.8875	23	−0.53	0.5178	−0.1343
8	1.	0.5864	−0.8499	24	−1.	0.4445	−0.0964
9	1.	0.6597	−0.8070	25	−1.	0.3712	−0.0639
10	0.78	0.7168	−0.7587	26	−1.	0.2979	−0.0367
11	0.16	0.7285	−0.7062	27	−1.	0.2216	−0.0149
12	−0.26	0.7095	−0.6429	28	−1.	0.1513	0.00015
13	−0.25	0.6915	−0.5909	29	−1.	0.0770	0.00012
14	−0.2 .	0.6737	−0.5403	30	−1.	0.0037	0.00018
15	−0.24	0.6720	−0.4910				

Table 2

The solution of RRA 1 model for MER problem, by Beale's Q.P. Algorithm

i	$u(i)$	$x_2(i)$	x_1	i	$u(i)$	$x_2(i)$	$x_1(i)$
0		0.	−1.				
1	0.50	0.0007	−1.	16	0.25	0.6525	−0.5480
2	0.54	0.0440	−0.9984	17	0.21	0.6436	−0.5004
3	0.58	0.0987	−0.9932	18	0.17	0.6274	−0.4537
4	0.61	0.1630	−0.9937	19	0.15	0.6078	−0.4084
5	0.62	0.2340	−0.9691	20	0.15	0.5897	−0.3645
6	0.61	0.3077	−0.9493	21	0.15	0.5768	−0.3218
7	0.59	0.3793	−0.9240	22	0.15	0.5698	−0.2798
8	0.56	0.4446	−0.8938	23	0.13	0.5648	−0.2382
9	0.52	0.5003	−0.8591	24	0.08	0.5541	−0.1971
10	0.47	0.5455	−0.8206	25	0.01	0.5296	−0.1572
11	0.43	0.5809	−0.7793	26	−0.08	0.4858	−0.1199
12	0.40	0.6083	−0.7356	27	−0.15	0.4236	−0.8648
13	0.37	0.6293	−0.6902	28	−0.19	0.3510	−0.5805
14	0.33	0.6444	−0.6435	29	−0.17	0.2822	−0.3490
15	0.29	0.6526	−0.5959	30	−0.10	0.2326	−0.1618

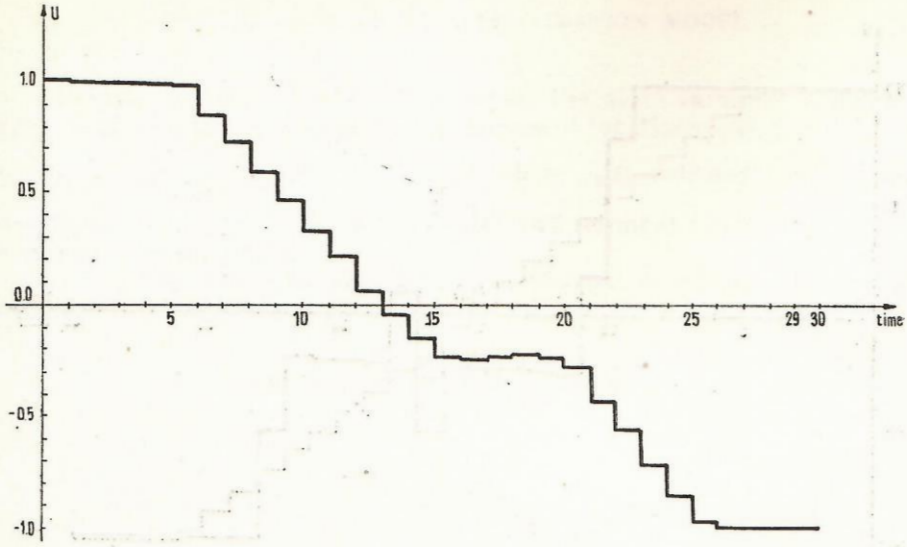

Fig. 1: MER optimal control solution obtained by solving M_2 model by SUMT (DFP and S—N's exponential penalty)

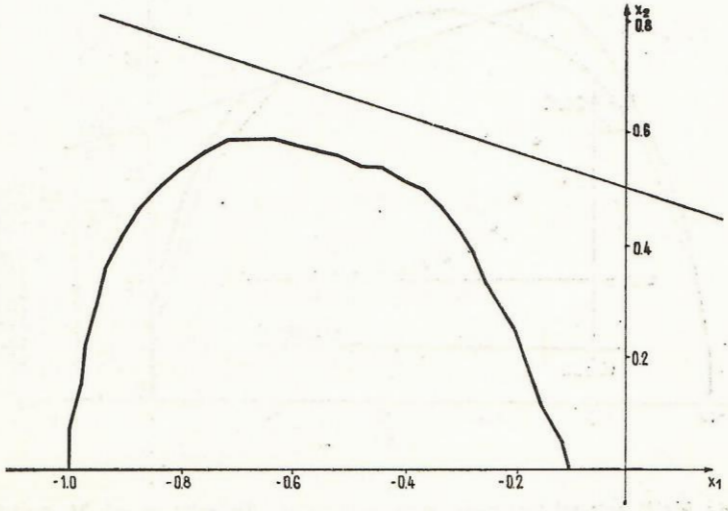

Fig. 2: MER optimal trajectory corresponding to the solution of M_2 model by SUMT.

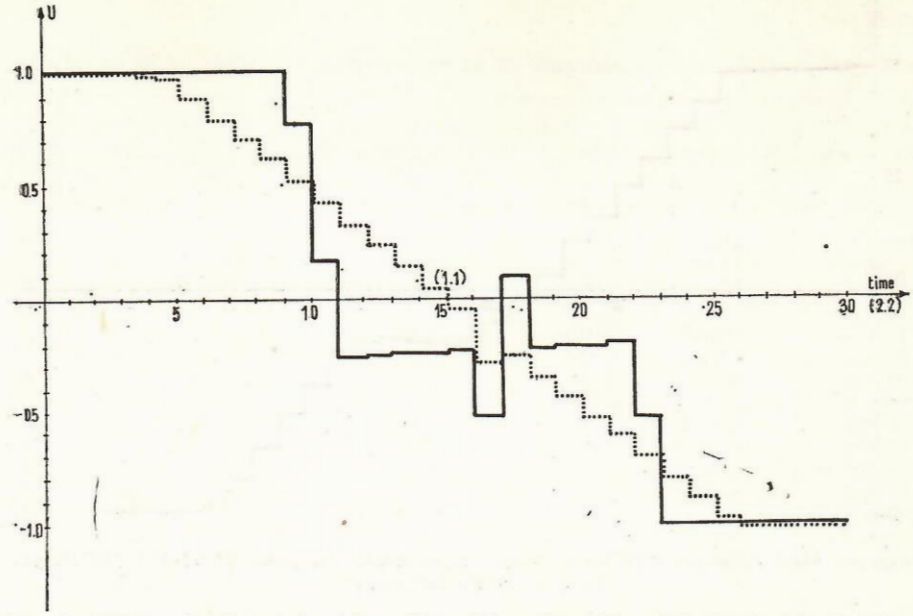

Fig. 3: MER optimal control solution obtained by solving M model with Beale's algorithm. Dot line represents the control of unconstrained problem.

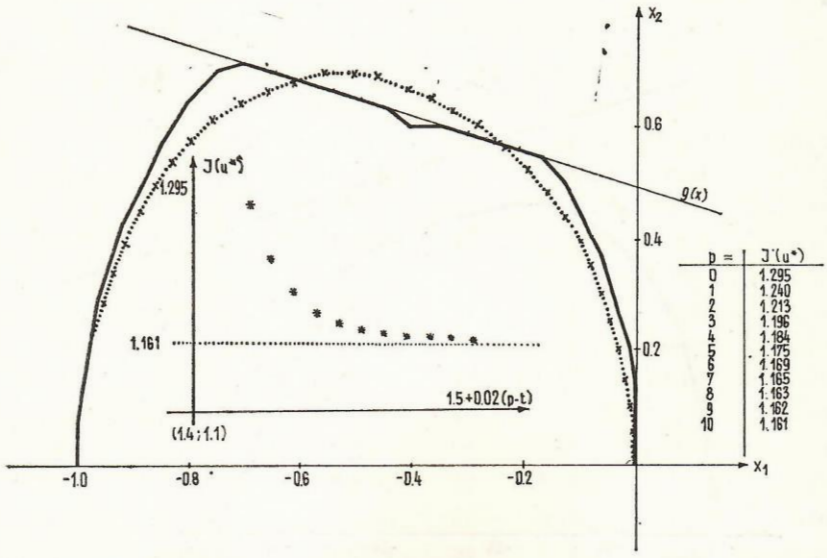

Fig. 4: MER optimal trajectory corresponding to the solution of M model by Beale's algorithm, by dot curve the optimal trajectory of unconstrained problem. The interior graph shows the variation of $J(u^*)$ function of the intercept to x_2 axis.

In order to apply the RR technique, the state variables and control are expanded in terms of an infinite sequence of functions $p_j(t)$, $j = 0, 1, ...$, so that $x_i = p_0(t) + \sum_{j=1}^{\infty} x_{ij} p_j(t)$, $i = 1, ..., n$ where $p_0(t)$ satisfies the boundary conditions and $p_j(t)$ vanish at the initial and terminal times. The $p_j(t)$ can be choosen as $sin(j\pi t/t_f)$.

Using the following substitution, restricted to r terms, for instance seven, of Rayleigh-Ritz expansion,

$$x_1 = -\cos\frac{\pi}{2}\frac{t}{tf} + \sum_{j=1}^{r} x_{1j} \sin\frac{j\pi t}{tf}$$

$$x_2 = \sum_{j=1}^{r} x_{2j} \sin\frac{j\pi t}{tf}$$

$$u = \sum_{j=1}^{r} x_{3j} \sin\frac{j\pi t}{tf}$$

— the Rayleigh-Ritz Approximation Model RRAM for the minimum energy regulator problem is:

$$min\ \Delta t \sum_{i=1}^{m} u_i^2 = \Delta t \sum_{i=1}^{m} \left(\sum_{j=1}^{r} x_{3j} \sin\frac{j\pi(i-1)\Delta t}{tf} \right)^2$$

subject to:

$$\dot{x}_1(i-1) = \frac{\pi}{2tf} \sin\frac{\pi}{2}\frac{(i-1)\Delta t}{tf} + \sum_{j=1}^{r} x_{1j}\frac{j\pi}{tf}\cos\frac{j\Pi\,(i-1)\Delta t}{tf} =$$

$$= \sum_{j=1}^{r} x_{1j} \sin\frac{j\pi(i-1)\Delta t}{tf} = x_2, \quad i = 1, 2, ..., m+1$$

$$x_2(i-1) = \sum_{j=1}^{r} x_{2j}\frac{j\pi}{tf}\cos\frac{j\pi(i-1)\Delta t}{tf} = \sum_{j=1}^{r} x_{3j} \sin\frac{j\pi(i)\Delta t}{tf} = u(i)$$

$$i = 1, 2, ..., m$$

$$\sum_{j=1}^{r} x_{1j} \sin\frac{j\pi(i-1)\Delta t}{tf} + 3\sum_{j=1}^{r} x_{2j} \sin\frac{j\pi\Delta t(i-1)}{tf} + 0.02(i-1)\Delta t \leqslant$$

$$\leqslant 1 \cdot 5 + \cos\frac{\pi}{2}\frac{(i-1)\Delta t}{tf} \qquad i = 1, 2, .., m$$

$$-1 \leqslant \sum_{j=1}^{r} x_{3j} \sin\frac{j\pi(i)\Delta t}{tf} \leqslant 1 \qquad i = 1, 2, ..., m.$$

The model can be simplified even more. Instead of choosing x_1, x_2 and u as independent variables we can choose only $x_1(i)$ as:

$$x_1(i-1) = -\cos\frac{\pi}{2}\,\frac{(i-1)\Delta t}{tf} + \sum_{j=1}^{r} x_{1j}\sin\frac{j\pi(i-1)\Delta t}{tf},\quad i = 1, \ldots, m+1$$

and as a consequence of dynamic conditions we have:

$$x_2(i-1) = \dot{x}_1(i-1) = \frac{\pi}{2t_f}\sin\frac{\pi}{2}\,\frac{(i-1)\Delta t}{t_f} + \sum_{j=1}^{r} x_{1j}\,\frac{j\pi}{t_f}\cos\frac{\pi j(i-1)\Delta t}{t_f}$$

$$i = 1, \ldots, m+1$$

$$u(i) = \dot{x}_2 = \ddot{x}_1 = \frac{\pi^2}{4t_f}\cos\frac{\pi(i-1)\Delta t}{2t_f} - \sum_{j=1}^{r} x_{1j}\left(\frac{j\pi}{t_f}\right)^2 \cdot \sin\frac{j\pi(i-1)\Delta t}{t_f}$$

$$i = 1, \ldots, m$$

Now as the system of restrictions we have the boundary conditions on x_2 and the algebraic ones, and dropping the first index, we have the RRA1 model:

$$\min u^2 = \Delta t\sum_{i=1}^{m} u^2(i)$$

$$E\quad \begin{cases} \sum_{j=1}^{r} \dfrac{j\pi}{t_f}\, x_j = 0 \\ \sum_{j=1}^{r} (-1)^j \dfrac{j\pi}{t_f}\, x_j + \dfrac{\pi}{2t_f} = 0 \end{cases} \qquad \text{from } x_2(0) = x_2(t_f) = 0$$

$$I\quad \begin{cases} \sum_{j=1}^{r}\left[\sin\dfrac{j\pi(i-1)\Delta t}{t_f} + \dfrac{3j\pi}{t_f}\cos\dfrac{j\pi(i-1)\Delta t}{t_f}\right] x_j \leqslant 0.02\,(i-1)\Delta t + \\ \qquad + \cos\dfrac{\pi(i-1)\Delta t}{2t_f} - 3\,\dfrac{\pi}{2t_f}\sin\dfrac{\pi(i-1)\Delta t}{2t_f} + 1.5 \\ \qquad i = 2, \ldots, m+1, \text{ from } g(x,t) \leqslant 0 \\ -1 - \left(\dfrac{\pi}{2t_f}\right)^2 \cos\dfrac{\pi(i-1)\Delta t}{2t_f} \leqslant \sum_{j=1}^{r} x_j \left(\dfrac{j\pi}{t_f}\right)^2 \sin\dfrac{j\pi(i-1)\Delta t}{t_f} \leqslant 1 - \\ \qquad - \left(\dfrac{\pi}{2t_f}\right)^2 \cos\dfrac{\pi(i-1)\Delta t}{2t_f} \\ \qquad i = 2, \ldots, m+1, \text{ from } -1 \leqslant u_i \leqslant 1 \end{cases}$$

which is a quadratic problem with r variables and $3m + 2$ constraints.

The model RRA1 has been solved, after introducing the nonnegative variables $x_j = x'_j - x''_{jj}$, by Beale's method for Q.P. for $r = 7$ and $m = 30$, i.e. solving a Q.P. problem with 14 variables and 90 constraints. The result of table 2 is graphed [—] in Fig. 5 and 6.

If some constraints are violated, as a consequence of the approximation we have considered, one has to use Balakrishnan's technique.

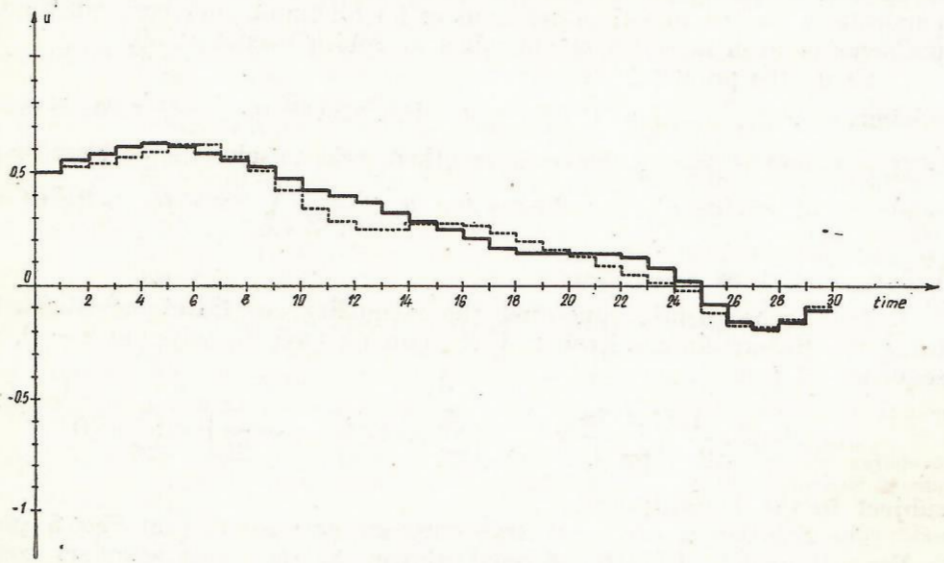

Fig. 5: MER optimal control solution.

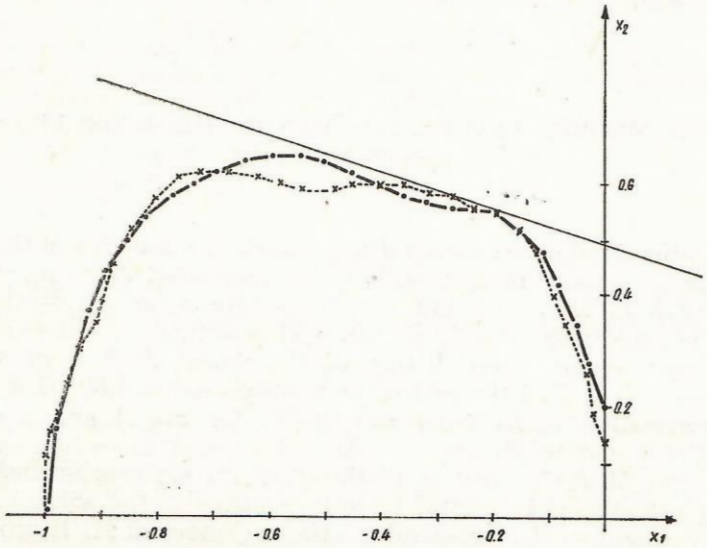

Fig. 6: MER optimal trajectory.

THE BALAKRISHNAN' S EPSILON TECHNIQUE

As it is known the Balakrishnan's Epsilon Technique [2] is a method that produces for the optimal control problem a sequence of trajectories and control that can be made arbitrarily close to the true optimum trajectory as epsilon, a penalty parameter, approaches zero. The method is very useful particularly for free and time problems or for minimum time cost functional problems or even more important when no solution exist at all.

Given the problem:

minimize $J(x,\ u,\ t) = \int_{t_0}^{t_f} L(x,\ u,\ t)\ \mathrm{d}t$, subject to $\dot{x} = f(x,\ u,\ t)$ and $x(t_0) = x_0$ and $x(t_f) = x_f$, the spsilon method seeks to solve the problem by a sequence of minimization problems $J(x,\ u,\ t) = \dfrac{1}{2\,\varepsilon}\int_{t_0}^{t_f} \| x - f(x, u, t) \|^2\, \mathrm{d}t +$ $\int_{t_0}^{t_f} L(x,\ u,\ t)\ \mathrm{d}t$, as $\varepsilon \to 0$.

As a consequence, supposing the inequality constraints are verified, using the Balakrishnan's Epsilon Technique we have to solve, as $\varepsilon \to 0$, a sequence of problems:

$$min\, u^2(\varepsilon) = \frac{1}{2\varepsilon}\left[\left(\sum_{j=1}^{r} \frac{j\pi}{t_f}\, x_j\right)^2 + \left(\sum_{j=1}^{r} (-1)^i \frac{j\pi}{t_f}\, x_j + \frac{\pi}{2t_f}\right)^2 + \sum_{i=1}^{m} u^2(i)\right.$$

subject to the I constraints.

The Solution control and trajectory are graphed (...) in Figs 5 and 6. For both models, RRA 1 and Balakrishnan, the right side boundary condition, probably due to the number of terms in the R—R—expansion, is not accurately satisfied.

THE OPTIMAL MINIMUM ENERGY, FUNCTION OF THE INTERCEPT OF $G(X,\ T)$ CONSTRAINT

The optimal minimum energy $J(u^*)$ is obvious a function of the intercept of the plane $g(x,\ t) = 0$ to x_2 axis. We have computed $J(u^*,\ p)$, function of a parameter p in the constraint $g(x,\ t)$, i.e. $g(x,\ t,\ p) = x_1 + 3x_2 + 0 \cdot 02\ (t - p) - 1 \cdot 5 \leqslant o$, for $p = 0,\ 1,\ ...,\ 6$ and $p = 50$.

The least square curve fitting of the model $J(u^*,\ t,\ p) = a[1 \cdot 5 + + 0 \cdot 02(p - t)]^{-\alpha} + b$ on the first seven value gives $a = 1.69197, b = 0.47255$ and the correlation coefficient $R = 0.94401$, for $\alpha = 2$ and $a = 1.72057$, $b = 0.71439$, $R = 0.94968$, for $\alpha = 3$.

The value of $J(u^*)$ function of the intercept are graphed in Fig. 4. On the same figure the dot optimal trajectory curve is the solution obtained for $p = 50$ and in fact it corresponds to the trajectory of MER problem with the additional constraint $g(x)$ being innactive.

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SUR LA RESOLUTION DU PROBLEME DU REGULATEUR D'ENERGIE MINIME A PROGRAMMATION MATHEMATIQUE

Dans le présent article les auteurs complètent les résultats antérieurs concernant la résolution du problème du régulateur d'énergie minime comme un problème de programmation mathématique .On présente les résultats concernant l'utilisation de l'algorithme de BEALE dans la résolution du modèle de programmation quadratique obtenu tant par une discrétisation du premier ordre que par celui obtenu par une approximation RAYLEIGH-RITZ de la solution. On présente aussi le modèle adéquat en vue de l'utilisation de la technique de Balakrishnan.

РАЗРЕШЕНИЕ ЗАДАЧИ РЕГУЛЯТОРА МИНИМАЛЬНОЙ ЭНЕРГИИ ПРИ ПОМОЩИ МАТЕМАТИЧЕСКОГО ПРОГРАММИРОВАНИЯ

В настоящей работе автор приводит новые результаты предыдущим результатам касающимся разрешения задачи регулятора минимальной энергии при помощи математического программирования. Приведены результаты, относящиеся к использованию алгоритма BEALE при разрешении модели квадратического программирования, полученной как через дискретизирование первого порядка, так и через аппроксимацию типа RAYLEIGH RITZ решения задачи. Приведена также соответствующая модель с целью применения приёма Балакришнана.

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