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Application of Benders' Partitioning Procedure to Solving a Total Model of Operative Production Prgramming

The application of Benders' partitioning procedure is proposed for the case wherein:

— some of the problem's basical constraints are generated during iterations;

— the object function is quasi-concave (discrete) and defined through brackets.

The study is based upon an actual study of smelting special steels and their heat-treatment.

I. INTRODUCTION

The determination and updating of the short-term fabrication programmes, on taking into account the technological conditions, stocks, resources and various economic aspects, is an important and complex problem with which the industrial enterprises are permanently confronted.

This problem was approached by some authors [1], [2], [3], with mathematical programming models, based upon the modelling of the studied process in its bottlenecks. If the process presents several bottlenecks, models for each separate point are constructed. The successive solving of the resulted models does not lead always to the best solutions, whereas their treating as a whole presents important difficulties, due to the increase of the dimension, and to the nature of total models.

Our study is based upon an actual process [4] of smelting special steels and of their heat-treatment, those being also the bottlenecks, whereas the stages of burden preparation, the aging, stripping and adjusting present a more reduced interest from the mathematical modelling viewpoint.

This work describes merely the link restrictions between the independent models of bottlenecks and the objective function of the total model. The concise description of the process, as well as independent modelling of the smelting and heat-treatment stages are found in [3].

For the total model obtained, the authors apply a more specialized Benders' partitioning procedure [5], [12] for: a) the case where in the link constraints are generated during Benders' iterations; b) the case of the non-linear (quasi-concave) objective functions, defined through brackets.

We mention that, though Benders' partitioning procedure is ingenious and very appreciated in the literature, it is relatively little experimented so far on concrete problems. The results of some encouraging applications are given in [6], [7], [8]. The revised Benders' procedure, generated by a practical application, is given in [11].

II. TOTAL MODEL DESCRIPTION

The model is composed of the steel's smelting stage model and of the heat-treatment stage model [4]. The smelting model has constraints of plan, raw materials, capacity and batch, presented by the end-user departments:

$$A_1 X \leqslant a_1, \quad X = (x_{jt}). \tag{1}$$

The heat treatment model contains only constraints of capacity and of treated batch requirements:

$$A_2 Y \leqslant a_2, \quad Y = (y_{jt}). \tag{2}$$

The variables used in the model are as follows:
x_{jt} — number of j — grade batches, smelted in period t;
y_{jt} — number of j — grade batches, heat treated in period t.

We mention that the same steel grade, smelted in distinct furnaces and/or poured from different steel moulds has various order numbers j, so that we have totally M grades, $j = 1, 2, ..., M$.

It is obvious that, in a total model, the variables x_{jt} and y_{jt} are interconditioned. That is to say, a grade $-j$ steel smelted in time period t can be heat-treated only in the next periods:

$$x_{jt} \geqslant y_{jt+1} + y_{jt+2} + \ldots + y_{jN}, \quad t = 1, \ldots N-1, \quad \forall_j \tag{3}$$

$$0 = y_{j1} + y_{j2} + \ldots + y_{jt}. \tag{3'}$$

For a grade $-j$ steel, smelted in period t, $N-2$ constraints must be redundant, from the $N-1$ ones. This can be achieved with the help of some variables $w_{jt} \in \{0,1\}$ and with constraints

$$\begin{gathered} -x_{jt} + \sum_{k=t+1}^{N} y_{jk} - dw_{jt} \leqslant 0 \quad t = 1, \ldots N-1, \quad \forall_j \\ \sum_{k=1}^{N-1} w_{jk} = N-2 \quad (\forall)_j \end{gathered} \tag{4}$$

$$\sum_{k=1}^{t} y_{jk} - ew_{jt} \leqslant 0 \qquad (4')$$

where d and e is a constant which makes constraints to be redundant.

For the j — grade steel to be heat-treated in period t, the steel grade must be smelted in one of the previous periods, a fact that leads to constraint analogous to (3), (3'), (4), (4'). Constraints (4) (4') are writtem under matrix form:

$$-B_1X + B_2Y - B_3W \leqslant 0.$$

We express the link between variables x_{jt} and y_{jt} also through the objective-function of the total model. To this end, let $\bar{t}_j$ be the period wherein grade-j steel is required in an end-user division. The coefficients of the objective-function are as follows:

$$c''_{jt}(\bar{t}_j) = c''_j(\bar{t}, \bar{t}_j) = \begin{cases} \dfrac{\alpha_{jt}}{\bar{t}_j - t} & t < \bar{t}_j \\ 1{,}5\ \alpha_{jt} & t = \bar{t}_j \\ \dfrac{\alpha_{jt}}{p_1 + p_2(t - \bar{t}_j)} & t > \bar{t}_j \end{cases} \qquad (5)$$

where p_1 and p_2 are penalty constraints due for exceeding the delivery plan or the proportional increase of the penalty as against the delay ($p_1 \geqslant p_2 \geqslant 1$), respectively. Coefficients α_{jt} are external adjustment coefficients [3], and here they are taken equal to 1.

The total model has the form:

$$\begin{gathered} A_1X \leqslant a_1 \\ A_2Y \leqslant a_2 \\ A_3W = a_3 \\ -B_1X + B_2Y - B_3W \leqslant 0. \quad X, Y \geqslant 0, \text{ integer}, \quad W \in \{0,1\} \\ max\ C'(Y)X + C''Y \end{gathered} \qquad (6)$$

where the elements of vector $C'(Y)$ are determined according to the same principle as in (5) with $\bar{t}_j = \begin{cases} t \text{ if exists } y_{it} \neq 0 \\ \text{N}+1 \text{ if not exists } y_{jt} \neq 0 \end{cases}$

hence the period t wherein grade-j steel is heat-treated.

III. TOTAL MODEL SOLVING METHOD

The total model (6) is non-linear in integer variables of large dimensions, wherein the non-linearity is given through bracket (5).

The possible direct solving methods would be the explicite Korte's method [9] or Geoffrion's method [10] as a rule. We shall try here to apply the Benders' partitioning procedure, neglecting the character of integers of some variables.

The function which introduces the non-linearity in the model is $C_j(t, \bar{t}_j)$ given through (5). We consider further on j as fixed.

We notice that $C(t,\bar{t})$ and $C(\bar{t}, t)$ are symmetrical as against axis $t = \bar{t}$.

Function $C(t, \bar{t})$ is neither concave, nor convex. As a matter of fact, this may be immediately verified, if we consider $t_1 = \bar{t} - 1$ and $t_2 = \bar{t} + 2$ and $\lambda = \frac{1}{3}$ and $\lambda = \frac{2}{3}$.

Function $C(t, \bar{t})$ is quasi-concave, due to the checking of inequality

$$C(\lambda t_1 + (1 - \lambda)t_2, \bar{t}) \geqslant min\,(C(t_1, \bar{t}), C(t_2, \bar{t}))$$

for the three cases $t < \bar{t}$, $t = \bar{t}$, $t > \bar{t}$, where $t = \lambda t_1 + (1 - \lambda)t_2$.

Continuing, let us consider the problems:

$$\begin{gathered} A_2Y \leqslant a_2 \\ A_3W = a_3 \\ B_2Y - B_3W \leqslant B_1X^\circ,\ Y \geqslant 0,\ W \in \{0,1\} \\ max\ C''Y \end{gathered} \tag{7}$$

$$\begin{gathered} A_2Y \leqslant a_2 \\ B_2'Y \leqslant B_1'X^\circ,\ Y \geqslant 0 \\ max\ C''Y \end{gathered} \tag{8}$$

where B_2', B_1' are submatrices from B_2 and B_1 of the non-redundant constraints corresponding to solution X°, a reason for which (7) and (8) have the same optimal solutions.

For the same reasons, the following problems have the same optimal solution

$$\begin{gathered} A_1X \leqslant a_1 \\ -B_1X \leqslant -B_2Y^\circ + B_3W^\circ \\ X \geqslant 0 \\ max\ C'(Y^\circ)\ X \end{gathered} \tag{9}$$

$$\begin{gathered} A_1X \leqslant a_1 \\ -X_1 \leqslant -B_2'Y^\circ \\ X \geqslant 0 \\ max\ C'(Y^\circ)X. \end{gathered} \tag{10}$$

On the basis of these remarks we give further down a specialization of Benders' partitioning procedure.

We consider Y° as arbitrary, so as to solve the problem:

$$A_1X \leqslant a_1$$

$$X \geqslant 0$$

$$C''Y^\circ + maxC'(Y^\circ)X \tag{11}$$

thus obtaining X° and information concerning the construction of submatrices B_1', B_2', so as to consider:

$$z = C'(Y^\circ)X^\circ + maxC''Y$$

$$A_2Y \leqslant a_2$$

$$B_2'Y \leqslant B_1'X^\circ,$$

$$Y \geqslant 0 \tag{12}$$

or its dual problem:

$$z = C'(Y^\circ)X^\circ + min\,(u_1, u_2)\begin{pmatrix} a_2 \\ B_1'X^\circ \end{pmatrix}$$

$$(u_1u_2)\begin{pmatrix} A_2 \\ B_2' \end{pmatrix} \geqslant C'', \quad u_1 \geqslant 0, \quad u_2 \geqslant 0. \tag{13}$$

Benders' "master" problem, with his optimal solution (12) or (13), respectively, is

$$\bar{z} = max\ Z$$

$$\leqslant C'(Y^1)X + (u_1^1,\ u_2^1)\begin{pmatrix} a_2 \\ B_1'X \end{pmatrix} \tag{14}$$

$$A_1X \leqslant a_1$$

$$-B_1'X \leqslant -B_2'Y^1, \quad X \geqslant 0.$$

Solution X^1 of problem (14) is used to construct problems (12) and (13), respectively.

Thus, the problem was solved through a specialization of Benders' procedure, which has the following advantages:

— the non-linearity through brackets disappears,

— the model's dimension becomes more convenient through generation of some constraints during Benders' iterations.

The convergence of the algorithm given this way does not suffer, as problems (7) and (8), or (9) and (10), respectively, have the same optimal solution, due to the construction of the non-redundant constraints only.

The obtained solution has several integer components, if the supplies, the plan and the terms are correlated.

The total model considered for the numeral experience, has the following features:

No. of furnaces		No. of periods	Period duration (days)	No. of steel groups	No. of steel mould grades	No. of ferro-alloys
for smelting	for heat-treatment					
2	10	5	2	2	10	10

therefrom a model resulted, having the matrix dimension A_1 of 27×50 and A_2 of 15×15 and the number of generated constraints, 63. The initial vector Y° was fixed, having all its components equal to zero.

After solving problem (11), we found:

$$X^\circ = (x^0_{jt}) = \begin{bmatrix} 12 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6.25 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.75 & 0 & 0 & 8 & 2 & 8.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 2 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 2 & 0 & 4 \end{bmatrix}$$

with $C'(Y^\circ)X^\circ = 33.5$.

After solving problem (12), we found:

$$Y^1 = (y^1_{jt}) = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.75 & 3.86 & 0.66 & 8 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6.13 & 0 & 0 & 0 & 8.8 & 2 & 0 & 0 & 0 \\ 0 & 0 & 13.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $C''Y^1 = 62.76$ and $Z = 33.5 + 62.76 = 96.26$.

The solving of the "master" problem (14) lead to

$$X^1 = (x^1_{jt}) = \begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.66 & 0 \\ 6.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.33 & 2 \\ 1.75 & 3.86 & 0.66 & 8 & 2 & 0 & 0.11 & 0 & 0 & 2 \\ 0 & 6.13 & 3.3 & 0 & 0 & 9.58 & 1.89 & 2 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for which $\bar{Z} = 170.36$.

When solving a new problem (12) we obtained:

$$Y^2 = (y_{jt}^2) = \begin{bmatrix} 12.88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 3.86 & 0.66 & 8 & 1.28 & 0 & 0 & 0 & 0 & 4 \\ 0 & 5.72 & 0 & 0 & 0 & 9.57 & 2 & 0 & 0 & 0 \\ 0 & 0.41 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $C''Y^2 = 89.733$ and $Z = 80.57 + 89.73 = 170.20$
so that X^1, X^2 are the optimal solutions.

We notice that we determined the optimum from two iterations.

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APPLICATION DU PRINCIPE DE LA DECOMPOSITION DE BENDERS A LA RESOLUTION D'UN MODELE GLOBAL DE PROGRAMMATION OPERATIONNELLE DE LA PRODUCTION

On suppose l'application du principe de la décomposition de Benders dans le cas où:
— certaines restrictions de base du problème sont engendrées pendant les itérations;
— la fonction but est quasi-concave (discrète) et définie par l'accolade.

L'étude a pour base un processus réel d'élaboration des aciers spéciaux et leur traitement thermique.

ПРИМЕНЕНИЕ ПРИНЦИПА РАЗЛОЖЕНИЯ БЕНДЕРСА ПРИ РАЗРЕШЕНИИ ГЛОБАЛЬНОЙ МОДЕЛИ ОПЕРАТИВНОГО ПРОГРАММИРОВАНИЯ ПРОИЗВОДСТВО

Предполагается применение принципа разложения Бендерса в случае, когда:
— некоторые основные ограничения задачи генерируются по мере исполнения итераций;
— целевая функция является почти вокнутой (дискретной) функцией и определена по интервалам.

Проводимое исследование основивается на реальном процессе плавки специальных сталей и их термической обработки.

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