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# Sequences of Random Variables Almost Periodical in Probability

*This paper extends the notion of numerical, almost-periodical sequ — ences in the case of random variables sequences and the notion of sequences of random variables, almost-periodical in probability is obtained.*

Some properties of the random functions, almost-periodical in probability is studied in papers [1] and [2], starting from the classical notion of almost-periodical function.

Paper [3] shows the link between the numerical almost-periodical sequences and the almost-periodical functions.

Having as receiving basis the results from [1], [2] and [3], we try, in this paper, to extend the notion of almost-periodical numerical sequences to the random variable sequences.

1. We denote by $\{\Omega, K, P\}$ a probability space and by $\mathcal{L}(\Omega, K, P)$ the set of random variables defined on $\Omega$ with complex values. If $Z$ is the set of integers, we introduce the following definition:

**Definition 1.** The sequences of random variables $(f_n)_{n\in z}$, nearly overall finite, is almost-periodical in probability, if for any $\varepsilon > 0$, $\eta > 0$ there is a natural number $N(\varepsilon, \eta)$, such that in any $N(\varepsilon, \eta)$ consecutive integers there is $p \in Z$, with the property that

$$P(\{\omega: |f_{n+p}(\omega) - f_n(\omega)| \geqslant \varepsilon\}) \leqslant \eta$$

for all $n \in Z$.

The existence of the necessary and sufficient condition, that a random variables sequences be almost -periodical in probability, presents interest. The following proposition gives such a condition.

**Proposition 1.** The random variables sequences $(f_n)_{n\in Z}$ is almost-periodical in probability, iff a subsequence convergent in probability uniform as against $n$*) can be extracted from any sequence having the form $(f_{n+m_k})\ k\in N$

**Proof.** We shall use the method exposed in [3]. Let $(m_k)_{k\in N}$ be a sequence of integers and $\varepsilon > 0$, $\eta > 0$. If we suppose the random variables $(f_n)_{n\in z}$ almost-periodical in probability, it results that there is a $N(\varepsilon, \eta)$ natural number, having the property that any set of $N(\varepsilon, \eta)$ consecutive integers contains a number $p$, so that

$$P(\{\omega : |f_{n+p}\ (\omega) - f_n(\omega)| \geqslant \varepsilon\}) \leqslant \eta$$

for any $n\in Z$. Let us take for any $k\in N$ the interval $(m_k - N(\varepsilon, \eta), m_k)$ and $p_k \in (m_k - N(\varepsilon, \eta), m_k)$ so that

$$P(\{\omega : |f_{n+p}(\omega) - f_n(\omega)| \geqslant \varepsilon\}) \leqslant \eta$$

for any $n\in Z$. We denote $q_k = m_k - p_k$, $k\in N$. It is obvious that $0 \leqslant q_k \leqslant < N(\varepsilon, \eta)$. It results that sequence $(q_k)_{k\in N}$ has an infinity of equal terms. We have

$$|f_{n+m_k}(\omega) - f_{n+q_k}(\omega)| = |f_{n+m_k-p_k+p_k}(\omega) - f_{n+m_k-p_k}(\omega)|$$

hence

$$P(\{\omega : |f_{n+m_k}(\omega) - f_{n+q_k}(\omega)| \geqslant \varepsilon\}) =$$

$$= P(\{\omega : |f_{n+m_k-p_k+p_k}(\omega) - f_{n+m_k-p_k}(\omega)| \geqslant \varepsilon\}) \leqslant \eta.$$

We denote by $q(\varepsilon, \eta)$ the value of the equal terms from sequence $(q_k)_{k\in N}$, and by $(m_{1k})_{k\in N}$ the corresponding subsequence.
It results that we may write:

$$P(\{\omega : |f_{n+m_{1k}}(\omega) - f_{n+q(\varepsilon,\eta)}(\omega)| \geqslant \varepsilon\}) \leqslant \eta$$

for any $n \in Z$. Let us have now sequence $(\varepsilon_n)_{n\in N}$, $\varepsilon_n \searrow 0$ and $(\eta_n)_{n\in N}$, $\eta_n \searrow 0$. For $(\varepsilon_1, \eta_1)$ we determine as above $q(\varepsilon_1, q_1)$ and subsequence $(m_{1k})_{k\in N}$, so that:

$$P(\{\omega : |f_{n+m_{1k}}(\omega) - f_{n+q(\varepsilon_1,\eta_1)}(\varepsilon)| \geqslant \varepsilon_1\}) \leqslant \eta_1$$

for any $n\in Z$.

We consider a subsequence $(m_{1k})_{k\in N}$ and pair $(\varepsilon_2, \eta_2)$ and we find $q(\varepsilon_2, \eta_2)$ subsequence $(m_{2k})_{k\in N}$ so that

$$P(\{\omega : |f_{n+m2k}(\omega) - f_{n+q(\varepsilon_2,\eta_2)}(\omega)| > \varepsilon_2\}) \leqslant \eta_2.$$

Proceeding in this manner, let $(m_{ee})_{e\in N}$ be the diagonal subsequence. There is $k(\varepsilon, \eta)\in N$ so that $2\varepsilon_n \leqslant \varepsilon$, $2\eta_n \leqslant \eta$ if $n \geqslant K(\varepsilon, \eta)$. We have:

$$\{\omega : |f_{n+m_{rr}}(\omega) - f_{n+m_{ss}}(\omega)| \geqslant \varepsilon\} \subset \{\omega : |f_{n+m_{rr}}(\omega) - f_{n+q(\varepsilon_k,\eta_k)}(\omega)| \geqslant \varepsilon_k\} \cup$$

$$\cup \{\omega : |f_{n+m_{ss}}(\omega) - f_{n+q(\varepsilon_k,\eta_k)}(\omega)| \geqslant \varepsilon_k\},$$

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* sequence $(m_k)_{k\in N}$ is a sequence of integers.

Hence

$$P(\{\omega: |f_{n+m_r}(\omega) - f_{n+m_{ss}}(\omega)| \geqslant \varepsilon\}) \leqslant 2\eta_k \leqslant \eta.$$

It results that random variables sequence $(f_{n+m_{ee}})_{e \in N}$ is Cauchy in probability, hence convergent.

**Conversely.** We suppose that the random variables sequences $(f_n)_{n \in Z}$ has the property that, from any subsequence $(f_{n+m_k})_{k \in N}$ a sequence convergent in probability can be extracted, but this is not almost-periodical in probability. There is $\varepsilon_0 > 0, \eta_0 > 0$ so that, for any $N(\varepsilon_0, \eta_0)$ there is a set of $N(\varepsilon_0, \eta_0)$ consecutive integers, with the property that for any integer $q$ from this set, there is $n_q \in Z$, so that .

$$P(\{\omega : |f_{n_q}(\omega) - f_{n_q+q}(\omega)| \geqslant \varepsilon_0\}) > \eta_0.$$

For $N(\varepsilon_0, \eta_0) = 1$ we find thus $q_1 \in Z$. For $N(\varepsilon_0, \eta_0) = 2$ we find $q_2 \neq q_1$ etc. Hence for any $k \in N$, there is $q_k \in Z$ and $n_k \in Z$, with the property that

$$P(\{\omega : |f_{n_k+q_k}(\omega) - f_{n_k}(\omega)| \geqslant \varepsilon_0\}) > \eta_0.$$

It results that not a single subsequence convergent in probability can be extracted from the random variables sequences $(f_{n+q_k})_{k \in N}$.

**Remark.** If the random function $f: R \to \mathcal{L}(\Omega, K, P)$ is normal in probability, it results that the random variables sequences given by $q_n(\omega) = f(n, \omega)$ is almost-periodical in probability. We shall prove that any random variables sequence almost-periodical in probability, proceeds from a random function almost-periodical in probability, in sense of H. Bohr.

**Proposition 2.** If random variables sequence $(q_n)_{n \in Z}$ is almost-periodical in probability, there is a random function $f$, almost-periodical in probability in sense of H. Bohr, so that

$$g_n(\omega) = f(n, \omega), \quad n \in Z.$$

**Proof.** Let $(g_n)_{n \in Z}$ be a random variables sequence, almost-periodical in probability. Let us have $n \in Z$ and $n \leqslant t < n + 1$. We define

$$f(t, \omega) = g_n(\omega) + (t - n)(g_{n+1}(\omega) - g_n(\omega)).$$

We shall prove that random function $f$ is almost-periodical in probability, is sense of H. Bohr. Let $\varepsilon > 0$, $\eta > 0$. There is $N(\varepsilon, \eta)$ with the property that any set of $N(\varepsilon, \eta)$ consecutive integers contains an integer, so that

$$P(\{\omega : |g_n(\omega) - g_{n+p}(\omega)| \geqslant \varepsilon/3\}) \leqslant \eta/3$$

ifor any $n \in Z$. For $n \leqslant t < n + 1$ we have

$$n + p \leqslant t + p < n + p + 1 \text{ and } 0 \leqslant t - n < 1$$

As

$$|f(t+p, \omega) - f(t, \omega)| \leqslant |g_{n+p}(\omega) - g_n(\omega)| +$$
$$+ |g_{n+p+1}(\omega) - g_{n+1}(\omega)| + |g_{n+p}(\omega) - g_n(\omega)|$$

we shall have

$$P(\{\omega : |f(t+p,\ \omega) - f(t,\ \omega)| \geqslant \varepsilon\}) \leqslant P(\{\omega : |g_{n+p}(\omega) - g_n(\omega)| \geqslant \varepsilon/3\}) +$$
$$+ P(\{\omega : |g_{n+p+1}(\omega) - g_{n+1}(\omega)| \geqslant \varepsilon/3\}) + P(\{\omega : |g_{n+p}(\omega) -$$
$$- g_n(\omega)| \geqslant \varepsilon/3\}) \leqslant \eta/3 + \eta/3 + \eta/3 = \eta.$$

for any $n \in Z$.

It results thus that $p$ is $a - (\varepsilon, \eta)$, an almost-period in probability for the random function $f(t, \omega)$. We see hence that $l(\varepsilon, \eta) = N(\varepsilon, \eta)$ satisfies the definition almost-periodical in probability, in sense of H. Bohr. It remains to prove that the random function $f(t, \omega)$ is continuous in probability.

Let us have $t_0 \in R$. If $t_0 = n$, then $f(t_0, \omega) = g_n(\omega)$. We consider $t > t_0$, $n < t < n + 1$. We have

$$f(t, \omega) - f(t_0, \omega) = (t - n)\ (g_{n+1}(\omega) - g_n(\omega)$$

and

$$|f(t, \omega) - f(t_0, \omega)| < |t - n|\ (|g_{n+1}(\omega)| + |g_n(\omega)|).$$

Let us have $\varepsilon > 0$, $\eta > 0$. We may write:

$$\{\omega : |f(t,\ \omega) - f(t_0, \omega)| \geqslant \varepsilon\} \subset \left\{\omega : |g_{n+1}(\omega)| \geqslant \frac{\varepsilon}{2|t - n|}\right\} \cup$$

$$\cup \left\{\omega : |g_n(\omega)| \geqslant \frac{\varepsilon}{2|t - n|}\right\}.$$

There is $\delta(\varepsilon, \eta) > 0$, so that, if $|t - n| = |t - t_0| < \delta(\varepsilon, \eta)$ we have

$$P\left\{\left(\omega : |g_{n+1}(\omega)| \geqslant \frac{\varepsilon}{2|t - n|}\right)\right\} \leqslant \eta/2 \quad \text{and} \quad P\left\{\left(\omega : |g_n(\omega)| \geqslant \frac{\varepsilon}{2|t - n|}\right)\right\} \leqslant \eta/2.$$

Finally,

$$P(\{\omega : |f(t, \omega) - f(t_0, \omega)| \geqslant \varepsilon\}) < \eta$$

where $t_0 = n$ and $!\,|t - t_0| < \delta\ (\varepsilon, \eta)$.

Let us consider $t < n$, $n - 1 \leqslant t < n = t_0$. Then,

$$f(t, \omega) - f(t_0, \omega) = g_{n-1}(\omega) + (g_n(\omega) - g_{n-1}(\omega))\,(t - n + 1) -$$
$$- g_n(\omega) = (g_n(\omega) - g_{n-1}(\omega))\,(t - n).$$

Similarly we show that $P(\{\omega : |f(t, \omega) - f(t_0, \omega)| \geqslant \varepsilon\}) \leqslant \eta$ as soon as $|t - t_0| < \delta_1(\varepsilon, \eta)$. We proved hence that the random function $f(t, \omega)$ is continuous in probability in point $t_0 = n$. If $n < t' < n + 1$ we can write: $f(t, \omega) - f(t', \omega) = g_n(\omega) + (t - t')\ (g_{n+1}(\omega) - g_n(\omega)$. Hence:

$$P(\{\omega : |f(t,\omega) - f(t', \omega)| \geqslant \varepsilon\}) \leqslant P(\{\omega : |t - t'|\,|g_{n+1}(\omega)| \geqslant \varepsilon/2\}) +$$
$$+ P(\{\omega : |t - t'|\,|g_n(\omega)| \geqslant \varepsilon/2\}).$$

There is $\delta(\varepsilon, \eta) > 0$ so that, if $|t - t'| < \delta$

we have

$$P(\{\omega: |t - t'|| g_{n+1}(\omega)| \geqslant \varepsilon/2\}) \leqslant \eta/2$$

and

$$P(\{\omega: |t - t'|| g_n(\omega)| \geqslant \varepsilon/2\}) \leqslant \eta/2.$$

Hence,

$$P(\{\omega: |f(t, \omega) - f(t', \omega)| \geqslant \varepsilon\}) \leqslant \eta$$

if $|t - t'| < \delta(\varepsilon, \eta)$.

Hence it results that the random function $f(t, \omega)$ is continuous in probability on $R$. Similarly to the classical almost-periodical function, we can give a sufficient condition that a random function be almost-periodical in probability, in sense of H. Bohr, with the help of the random variables sequence, almos-periodical in probability.

**Proposition 3.** Let us have random function $f(t, \omega)$, uniformly continuous in probability on $R$. If there is a sequence $(\delta_k)_{k \in N}$ monotone, decreasing towards zero, with property that the random-variables sequence $g_n(\omega) = f(n\delta_k, \omega)$, $k \in N$, $n \in Z$ be almost-periodical in probability, then random function $f(t, \omega)$ is almost-periodical in probability in sense of H. Bohr.

**Proof.** Let us have $\varepsilon > 0$, $\eta > 0$ and $(\delta_k)_{k \in N}$ the sequence in the proposition. There exists $\delta(\varepsilon, \eta) > 0$ so that

$$P(\{\omega: |f(t', \omega) - f(t'', \omega)| \geqslant \varepsilon\}) \leqslant \eta, \text{ if } |t' - t''| < \delta.$$

Let $k(\varepsilon, \eta) \in N$ so that $\delta_k < \delta(\varepsilon, \eta)$ if $k \geqslant k(\varepsilon, \eta)$. We consider $k$ as fixed. It results that the random variables sequence $(g_n)_{n \in N}$ is almost-periodical in probability.

Let us have $N(\varepsilon, \eta) \in N$, with the property that any set of $N(\varepsilon, \eta)$ consecutive integers contains an integer $p$, so that

$$P(\{\omega: |g_{n+p}(\omega) - g_n(\omega)| \geqslant \varepsilon/3\}) \leqslant \eta/3.$$

Let us take $\tau = p\delta_k$. Let us have $t \in R$. There is $n \in Z$ so that $n\delta_k \leqslant t < (n + 1)\delta_k$. Then

$$P(\{\omega: |f(t + \tau, \omega) - f(t, \omega)| \geqslant \varepsilon\}) \leqslant P(\{\omega: |f(t + \tau, \omega) - f(n\delta_k + \tau, \omega)| \geqslant$$

$$\geqslant \varepsilon/3\}) + P(\{\omega: |f(n\delta_k + \tau, \omega) - f(n\delta_k, \omega)| \geqslant \varepsilon/3\}) + P(\{\omega: |f(n\delta_k, \omega) -$$

$$- f(t, \omega)| \geqslant \varepsilon/3\}) \leqslant \eta/3 + \eta/3 + \eta/3 = \eta.$$

Hence, $\tau = p\delta_k$ is a $-(\varepsilon, \eta)$ almost-period in probability. We can take $l(\varepsilon, \eta) = (N(\varepsilon, \eta) + 1)\delta_k$ and it results that the random function is almost-periodical in probability, in sense of H. Bohr.

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## SUITES DE VARIABLES ALEATOIRES PRESQUE PERIODIQUES EN PROBABILITE

Les auteurs étendent la notion de suite numérique, presque périodique, au cas des suites de variables aléatoires, obtenant la notion de suite de variables aléatoires presque périodique en probabilité.

## РЯДЫ АЛЛЕАТОРНЫХ ПЕРЕМЕННЫХ ПОЧТИ ПЕРИОДИЧНЫЕ ПО ВЕРОЯТНОСТИ

В этой статье понятие числового ряда, почти периодичного распространяется на случай рядов аллеаторных переменных, получая понятие ряда переменных, почти периодичного по вероятности.

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