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# Some Considerations on Stochastic Linear Programming

*In this paper a method is suggested in view of approximately determining the distribution function of the optimum of a problem of linear programming with random coefficients.*

In this paper, some remarks on the solution and optimum of stochastic linear programming problems are made. The notations are shown in paragraph 1. A problem regarding the convergence of solutions is stated in paragraph 2. The difficulty of exactly determining the distribution function of the optimum in a particular case is made to stand out in paragraph 3. A method for approximating the distribution function of the optimum is given in paragraph 4. It consists in approximating the coefficients by simple random variables. The same method is applied in paragraph 4 for solving systems of linear equations with aleatory coefficients.

1. Notations. Let be $A = (a_{ij})$, $1 \leqslant i \leqslant m$, $1 \leqslant j < n$, a matrix $m \times n$, $b = (b_k)$, $1 \leqslant k \leqslant m$, a matrix $m \times 1$, $c = (c_l)$, $1 \leqslant l \leqslant n$, a matrix $n \times 1$. A matrix $1 \times n$ will be noted by $x = (x_j)$. We shall assume that the matrices $A$, $b$, $c$, are random variables on a probability space $(\Omega, K, P)$. For every $\omega \in \Omega$ let us set

$$O_\omega = \{x\colon A(\omega)\ x \leqslant b(\omega), \quad x \geqslant 0\}$$

and let $x_0(\omega)$ be the optimum solution of the problem of linear programming which consists in making $c'(\omega)x$ maximum for $x \in O_\omega$. We note that we give the following sense to $x_0(\omega)$: $x_0(\omega)$ is a void set (in the case where the problem considered has no solution), a set formed of a single point in the $n$-dimensional

Euclidean space (in the case where the problem considered has a unique solution), or a convex linear variety in the $n$-dimensional Euclidean space (in the case where the problem considered does not have a unique solution).

Let $\lambda(\omega) = \sup_{x \in O_\omega} c'(\omega)\, x$

if $O_\omega^n \neq \emptyset$. Noting $\Omega_0 = \{\omega : O_\omega \neq \emptyset\}$, $K_0 = K_0 \cap \Omega_0$, $P_0 =$ the probability induced by $P$ on $K_0$ (i.e. $P_0(A) = \dfrac{P(A)}{P(\Omega)_0}$, $A \in K_0$) under the hypothesis that $P(\Omega_0) > 0$, then it is not difficult to see that $\lambda$ is a random variable in $(\Omega_0, K_0, P_0)$.

2. The optimum solution $x_0(\omega)$ offers a concrete example of a variable whose values are sets. It may therefore be stated that the study of this type of "random" variable (a proper definition of the term "random" in the general case still remains to be found) is naturally imperative. We shall notice that if $A, b, c$, are simple matricial random variables, then the same thing may be said of $x_0(\omega)$, i.e. the number of its values is finite. In other words, there will be a partition $\Omega = \Omega_1 \cup \ldots \cup \Omega_s$ such that $x_0(\omega) = x_0$ for $\omega \in \Omega_p$, $1 \leqslant p \leqslant s$. This idea has been used in [3] for the case where only $b$ and $c$ are random. The following problem seems to be an interesting one: if $b$, $c$, are not simple variables, they may be approximated by sequences of such variables. Correspondingly, we shall have sequences of optimum solutions. Is there any convergence property of these optimum solutions?

3. An important problem of the stochastic linear programming is the study of the random variable $\lambda$. It would be interesting, in the first place, to find its distribution. No general methods are known, so far, liable to lead *effectively* to the finding of this distribution (see [1]). The only explanation of this situation is the extreme complexity of the problem. This fact will be seen from the very simple example hereafter [2].

Let us place ourselves in the conditions: $m = n = 2$; the matrices $A$ and $b$ are constant and have the values:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$c_1$ and $c_2$ are independent normal random variables $N(m_1, \sigma_1)$ and $N(m_2, \sigma_2)$. Hence, for any $\omega \in \Omega$, the set $O_\omega$ coincides with the unit square, and

$$\lambda(\omega) = \max_{0 \leqslant x_1,\, x_2 \leqslant 1} [c_1(\omega)x_1 + c_2(\omega)x_2]$$

It is well known that the value $\lambda(\omega)$ is reached in one of the vertices of the unit square: $O(0, 0)$, $A(1, 0)$, $B(1, 2)$, $C(0,1)$. This occurs, respectively, in $O$, $A$, $B$ or $C$, as

$$\begin{cases} c_1(\omega) \leqslant 0 \\ c_2(\omega) \leqslant 0 \\ c_1(\omega) + c_2(\omega) \leqslant 0 \end{cases}$$

$$\begin{cases} c_1(\omega) \geqslant 0 \\ c_1(\omega) \geqslant c_1(\omega) + c_2(\omega) \\ c_1(\omega) \geqslant c_2(\omega) \end{cases}$$

$$\begin{cases} c_1(\omega) + c_2(\omega) \geqslant 0 \\ c_1(\omega) + c_2(\omega) \geqslant c_1(\omega) \\ c_1(\omega) + c_2(\omega) \geqslant c_2(\omega) \end{cases}$$

$$\begin{cases} c_2(\omega) \geqslant 0 \\ c_2(\omega) \geqslant c_1(\omega) \\ c_2(\omega) \geqslant c_1(\omega) + c_2(\omega) \end{cases}$$

Noting by $E_1$, $E_2$, $E_3$, $E_4$ the sets in $K$ defined by the above systems of inequalities, we shall have

$$\lambda(\omega) = \begin{cases} 0 & \text{if } \omega \in E_1 \\ c_1(\omega) & \text{if } \omega \in E_2 \\ c_1(\omega) + c_2(\omega) & \text{if } \omega \in E_3 \\ c_2(\omega) & \text{if } \omega \in E_4 \end{cases}$$

Hence

$$F(x) = P(\lambda(\omega) < x) = \sum_{r=1}^{4} P(\{\lambda(\omega) < x\} \cap E_r) = \delta_x([0, \infty])\ P(E_1) +$$

$$+ P(c_1(\omega) < x,\ c_1(\omega) \geqslant 0,\quad c_2(\omega) \leqslant 0,\ c_1(\omega) \geqslant c_2(\omega)) +$$

$$+ P(c_1(\omega) + c_2(\omega) < x,\ c_1(\omega) + c_2(\omega) \geqslant 0,\ c_2(\omega) \geqslant 0,$$

$$c_1(\omega) \geqslant 0) + P(c_2(\omega) < x,\ c_2(\omega) \geqslant 0,\ c_2(\omega) \geqslant c_1(\omega),\ c_1(\omega) \leqslant 0).$$

It can be seen that $F(x) = 0$ for $x < 0$.

For $x \geqslant 0$ we can write

$$F(x) = P(c_1(\omega) \leqslant 0,\ c_2(\omega) \leqslant 0) + P(0 \leqslant c_1(\omega) < x,\ c_2(\omega) \leqslant 0) +$$

$$+ P(c_1(\omega) + c_2(\omega) < x,\ c_1(\omega) \geqslant 0,\ c_2(\omega) \geqslant 0) +$$

$$+ P(0 \leqslant c_2(\omega) < x,\ c_1(\omega) \leqslant 0) = \Phi\left(-\frac{m_1}{\sigma_1}\right)\Phi\left(-\frac{m_2}{\sigma_2}\right) + \Phi\left(-\frac{m_2}{\sigma_2}\right)$$

$$\left[\Phi\left(\frac{x - m_1}{\sigma_1}\right) - \Phi\left(\frac{m_1}{\sigma_1}\right)\right] + \frac{1}{2\pi\sigma_1\sigma_2}\int_0^x \int_0^{x-u} \exp\left[-\frac{(u - m_1)^2}{2\sigma_1^2} -\right.$$

$$\left. - \frac{(v - m_2)^2}{2\sigma_2^2}\right] du\,dv + \Phi\left(-\frac{m_1}{\sigma_1}\right)\left[\Phi\left(\frac{x - m_2}{\sigma_2}\right) - \Phi\left(-\frac{m_2}{\sigma_2}\right)\right],$$

where $\Phi$ is Laplace's well-known function

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left(-\frac{x^2}{2}\right) dx$$

After elementary transformations we obtain ($x \geqslant 0$)

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_0^x \exp\left[-\frac{(u-m_1)^2}{2\sigma_1^2}\right] \Phi\left(\frac{x-u-m_2}{\sigma_2}\right) du + \Phi\left(-\frac{m_1}{\sigma_1}\right)\Phi\left(\frac{x-m_2}{\sigma_2}\right)$$

The result is obtained after simple, but long computations.

We shall hereafter outline a possibility to solve the problem approximately in the general case.

4. Let us first assume that $A$, $b$, $c$, are simple matricial aleatory variables, in other words, there is a partition $\Omega_0 = \Omega_1 \cup \ldots \cup \Omega_s$ of $\Omega_0$, so that

$$A(\omega) = A^{(p)}, b(\omega) = b^{(p)}, \; c(\omega) = c^{(p)}$$

if $\omega \in \Omega_p$, where the matrices $A^{(p)}$, $b^{(p)}$, $c^{(p)}$, are constant, $1 \leqslant p \leqslant s$. In this case, $\lambda(\omega)$ for $\omega \in \Omega_p$ is determined by solving a non-stochastic linear programming problem, and thereafter, the determination of the distribution function of $\lambda$ no longer presents any difficulty (the solution even being exact).

In the general case $A$, $b$, $c$, can be approximated by simple matricial random variables which yield distribution functions of the maximum, according to the procedure described. It appears plausible that, under broad conditions, these distribution functions should converge to the distribution function sought. The problem has an immediate affirmative answer in case there is, in the $n$-dimensional space, a sphere which contains all the $O_\omega$, $\omega \in \Omega$ sets. The proof immediately results from considerations of continuity. The study of this problem would prove interesting in the general case.

5. The foregoing considerations lead us to some remarks on the distribution function of the solution of systems of linear equations with random coefficients.

Let

$$\sum_{1-q}^{s} \alpha_{pq} x_q = \beta_p, \qquad 1 \leqslant p \leqslant s$$

be a system of linear equations where $\alpha_{qp}$, $\beta_p$ are random variables on $(\Omega K, P)$. Let us assume that, almost certainly, $D = |\alpha_{qp}| \neq 0$. Applying Cramer's law we obtain

$$x_p = \frac{D_q}{D}, \quad 1 \leqslant q \leqslant s$$

$D_2$ having an obvious significance. If $\alpha_{qp}$ and $\beta_p$ are simple random variables, then the solution $x = (x_1, \ldots, x_s)$ will also be a simple random variable whose distribution function can be written without any difficulty. In the general case, $\alpha_{pq}$ and $\beta_p$ are approximated by simple variables. The formulae determining $x_q$ show that the approximate solution tends almost surely to the exact solution. A procedure results herefrom, for approximating the distribution function of the solution.

## REFERENCES


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## QUELQUES CONSIDERATIONS SUR LA PROGRAMMATION LINEAIRE STOCHASTIQUE


Dans cet article on propose une méthode pour la détermination de la fonction de répartition de l'optimum d'un problème de programmation linéaire à coefficients aléatoires. L'idée de la méthode est d'approximer les variables aléatoires par des variables aléatoires simples. On aborde également, de la même façon, le problème de la détermination de la fonction de répartition de la solution des systèmes linéaires d'équations.


## НЕСКОЛЬКО ЗАМЕЧАНИЙ В СВЯЗИ СО СЛУЧАЙНЫМ ЛИНЕЙНЫМ ПРОГРАММИРОВАНИЕМ


В работе излагается метод приблизительного определения функций распределения оптимума данной задачи линейного программирования со случайными коэффициентами. Метод основан на приближении случайных переменных при помощи простых случайных переменных. Наряду с этим, таким же образом рассматривается и вопрос определения функции расспределения решений линейных систем уравнений.