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Dynamics of Social Needs and Economic Growth

In the paper it is justified the necessity of including within the economic growth models of the elements describing the dynamics of economic needs. Further on, it is suggested a dynamic model of economic growth where the dynamics of social needs is also taken into consideration. The optimum conditions are stated for the proposed model and certain qualitative properties of the optimal trajectories are proved.

The identification, as precise as possible, of the laws governing the process of economic growth, represents a requirement of great present interest for the economic sciences. This refers, first of all, to the quantification, as exact as possible, of the real potential of growth of the economy, as a function of its main determining factors: the volume, structure and qualification level of the labour force, the volume and structure of fixed funds and of the natural resources which can be attracted in the economic circuit, the development level of scientific research, the capacity of the economy to assimilate the new results obtained in the domains of science and technique, a.s.o.

At the same time, in order to formulate the problem of economic growth as a problem of optimal programming, the fundamentation of the macroeconomic optimality criterion is required. This criterion is meant to introduce an ordering relation in the set of growth trajectories feasible for the system of the national economy.

It is well known that the fundamentation of the optimality criterion plays an important part in the construction of any optimal programming model, since the optimal solution of the problem is selected on its basis. This problem is of increased importance when it refers to the whole national economy, to the choice of the strategy of economic growth.

The transcription of the problem of economic growth in the terms of mathematical programming was realized as early as 1928 by the English mathematician and logician Frank Ramsey [11]. In his economic growth model, Ramsey chooses as optimality criterion the maximization of the following functional:

$$J = \int_0^\infty U(c_t)\, dt$$

where c_t stands for the volume of consumption allotted at moment t and $U(.)$ is the utility function of consumption (an increasing concave function).

Subsequently, one endeavoured to improve the macroeconomic optimality criterion. Thus, D. Cas (1966), T. Koopmans (1965), E. Malinvaud (1965), P. Samuelson (1965) consider as macroeconomic optimality criteria some functionals of the form

$$J = \int_0^\infty e^{-\rho t} U(c_t)\, dt$$

where c_t represents the volume of consumption per unity of labour force and ρ is the discount factor.

Afterwards, criteria of the form

$$J = \int_0^\infty e^{-\rho t} L_t\, U(c_t)\, dt$$

or

$$J = \int_0^\infty e^{-\rho t}\, P_t\, U(c_t)\, dt$$

have been considered, where L_t stands for the volume of labour force and P_t for the volume of population at moment t.

Leaving aside the difficulties connected with the identification of the concrete analytic form of the utility function $U(.)$, we have to mention that the above functionals are far off reflecting the macroeconomic optimality criterion. Nevertheless, the research performed till now in the field of modelling economic growth has the merit of allowing to obtain some very interesting results concerning the qualitative properties of the trajectories of economic growth. In this respect, we can mention, for instance, the „turnpike" properties connected with the "golden rule" of economic development.

A remarkable analysis of the criterion and of the forms of the macroeconomic optimum for the socialist economy has been performed by Prof. Em. Dobrescu [4]. Asserting the close connection between the macroeconomic optimality criterion and the fundamental economic law of a social system, Prof. Em. Dobrescu reaches the conclusion that this is identified with the fundamental goal of society. As concerns the socialist system, Em. Dobrescu considers that the fundamental law has to be formulated as the "Law of maximization of social welfare".

Reflecting the decisive part played by the augmentation of the welfare of the nation in the general strategy of economic-social development of the

country, in the general cybernetic system of the national economy, worked out by Acad. Prof. Manea Mănescu, the *system of the welfare of the nation* plays a central part. This allowed to investigate, by the aid of the methods and techniques specific to the economic cybernetics, numerous aspects concerning the mechanism of optimal, balanced and proportional economic growth.

The research performed so far in order to define the concepts of social welfare and quality of life, respectively, put into evidence the fact that this concept appears as a resultant of numerous social-political, cultural, biological, psychical, philosophical factors.

Thus, the volume, variety and quality of consumption goods and services for the population, the level of endowment of the population with lodgements and durable goods, the level of the incomes and the purchasing power of the population, the working conditions, the duration of the working week and working day, as well as other elements of the state and dynamics of the *economic conditions* represent important co-ordinates which define social welfare and the quality of life, respectively.

Many factors of the *social-political* and cultural spaces influence also the level of social welfare and quality of life, respectively: the level of development and the quality of education, of the culture, of health protection, the principles of socialist etics and equity and the way of their application, the degree of participation to social-political and decisional activities, the degree of realization of social and professional aspirations, the state of social security, of public order and of the justice, the quality of man-society relations and of inter-human relations, the access to information, the possibilities and ways of using the leisure time a.s.o.

Social welfare is also strongly influenced by the *characteristics of the man-made environment*: the degree of urbanization, the structure of the localities, the architecture and the level of systematization of the localities, the density of constructions and the share of verdure spots inside the localities, the quality and the level of household endowments, the conditions of sanitary engineering, the density, the state and the quality of the lines of communication, the conditions of tourism and leisure a.s.o. — as well as by the *characteristics of the natural environment.*

Together with the acceleration of the process of economic growth, of the industrialization, the problem of the pollution of the environment and that of the corresponding economic and social-political pollution control measures acquire a special importance as concerns social welfare.

Social welfare is also influenced by a number of demographic factors: the types and the ways of organization of the families, the dynamics of the population in urban-rural environments, the density of population in several geographic centers, the level and dynamics of the migration a.s.o.

The analysis of the modalities of action of different factors which influence social welfare has to be correlated with the analysis of the evolution laws of *social needs*, regarded as *requirements* or *aspirations*, respectively.

These are, in their turn, under the incidence of numerous factors studied by social-cultural antropology, which analyzes human behaviour in its social, economic and cultural contexts (ideals, traditions, customs, behaviour rules, disponibilities, moral and legal codes).

In the study of social welfare, as well as that of the mechanism of generation of the interests determined by them, the knowledge, as precise as possible, of the axiologic components and of the dynamics of the systems of values plays an important part.

1. THE NECESSITY OF INTRODUCING IN THE ECONOMIC GROWTH MODELS OF SOME ELEMENTS MEANT TO DESCRIBE THE DYNAMICS OF SOCIAL NEEDS

Social need represents a very complex economic-social category, the dynamics of which is determined by a multitude of factors of economic, social, cultural, psychological nature.

Though the knowledge of social needs and of the extent to which they are satisfied is necessary in order to transpose the Fundamental law of society in the terms of optimal programming [3], the majority of the mathematical models of economic growth do not contain any elements concerning these aspects.

The omission of the aspects concerning the dynamics of social needs can lead to incomplete or even false conclusions concerning the evolution laws of the system of the national economy.

In order to analyse the way in which some qualitative conclusions concerning the trajectories of economic growth can contradict the evolution laws of social needs, we shall refer to a "standard" model of economic growth with discrete variables, reduced to the "per capita" form.

The considered model is one of the best studied in the literature. It has the following form:

$$(I)\quad \begin{cases} k_{t+1} = gk_t + f(k_t) - c_t; \quad k_0 = \alpha\ ; \quad k_T \geqslant \omega \\ J_T(c) = \max\limits_{\underline{c} \leqslant c_t \leqslant f(k_t)} \sum\limits_{t=0}^{T-1} \delta^t\ U(\lambda c_t) \\ \underline{c} \leqslant c_t \leqslant f(k_t) \\ g < 1; \quad 0 < \delta \leqslant 1. \end{cases}$$

The notations are the following:

k_t — technical endowment of labour in year t (per capita capital);
$f(.)$ — the macroeconomic production function;
c_t — the per capita volume of the resources meant to satisfy the consumption needs;
$U(.)$ — the utiltiy function of consumption;
δ — discount factor;
ω — the minimum level which has to be reached by the technical endowment of labour at the end of the time horizon;
$\underline{c}$ — the minimum admitted level for the per capita consumption.

The dynamic model (I) contains a state variable k_t, and a control parameter c_t.

The sequence

$$c = (c_0,\ c_1,\ \ldots,\ c_{T-1})$$

is called *feasible control* if the solution of the equation

$$k_{t+1} = gk_t + f(k_t) - c_t; \quad k_0 = \alpha$$

satisfies the condition $k_T \geqslant \omega$. Moreover, conditions $\underline{c} \leqslant c_t \leqslant f(k_t)$ for $t = 0, 1, ..., T-1$ are to be satisfied.

The feasible control $\tilde{c}$ is an *optimal control* if it maximizes the functional:

$$\sum_{t=0}^{T-1} \delta^t U(\lambda c_t)$$

in the class of all feasible controls.

If functions $f(.)$ and $U(.)$ are concave and $\bar{k}_T > \omega$, where $\bar{k}_t$ is the solution of equation

$$\bar{k}_{t+1} = g\bar{k}_t + f(\bar{k}_t) - \underline{c}; \quad k_0 = \alpha$$

then the optimal solution exists and it is actually obtained by applying the Maximum principle for discrete systems [5].. In other words, the Maximum principle provides also sufficient conditions of optimality. In the case in which at least one of the functions $f(.)$ or $U(.)$ is strictly concave, the optimal solution is unique [5].

The Hamiltonian function of the system is the following:

$$H_t(k_t, c_t, \psi_t) = \psi_t[gk_t + f(k_t) - c_t] + \bar{\mu}_0 \delta^t U(\lambda c_t) + \\ + \mu_t[f(k_t) - c_t] + \mu_t^1[\tilde{c}_t - \underline{c}] \tag{1}$$

where the adjoint variable ψ_t satisfies the equation

$$\psi_{t-1} = \frac{\partial H_t(k_t, c_t, \psi_t)}{\partial k_t}. \tag{2}$$

In the case when $\bar{k}_T > \omega$, we have $\bar{\mu}_0 > 0$ and we can consider $\bar{\mu}_0 = 1$[5]

The necessary optimality condition, provided by the Maximum principle, is the following:

$$\frac{\partial H_t(\tilde{k}_t, \tilde{c}_t, \tilde{\psi}_t)}{\partial c_t}(c_t - \tilde{c}_t) \leqslant 0 \tag{3}$$

for any control $c_t \in [\underline{c}, f(\tilde{k}_t)]$ and any $t = 0, 1, ..., T-1$.

Moreover, the following transversality condition is satisfied:

$$\tilde{\psi}_{T-1}[\tilde{k}_T - \omega] = 0; \quad \tilde{\psi}_{T-1} \geqslant 0 \tag{4}$$

as well as the conditions:

$$\mu_t[f(\tilde{k}_t) - \tilde{c}_t] = 0, \quad t = 0, 1, ..., T-1. \tag{5}$$

$$\mu_t^1[\tilde{c}_t - \underline{c}] = 0.$$

We noted by $\tilde{c}_t$ the optimal control, and by $\tilde{k}_t$ and $\tilde{\psi}_t$ the corresponding values of the state variable and of the dual variable, respectively.

Noting

$$q_t = \frac{\psi_t}{\lambda \delta^t}, \tag{6}$$

in the case in which $\underline{c} < \tilde{c}_t < f(\tilde{k}_t)$, the optimality condition becomes

$$\tilde{q}_t = U'(\lambda \tilde{c}_t) \tag{7}$$

and the dynamic equation of the dual variable $\tilde{q}_t$ is the following:

$$\tilde{q}_{t-1} = \delta[g + f'(\tilde{k}_t)]\tilde{q}_t. \tag{8}$$

From (7) it follows:

$$\tilde{c}_t = \frac{1}{\lambda} W(\tilde{q}_t). \tag{9}$$

We noted by W(.) the inverse of the function $U'(.)$.

From relations (8) and (9) we obtain the dynamic equation of consumption:

$$\tilde{c}_t = \frac{1}{\lambda} W\left(\frac{U'(\lambda c_{t-1})}{\delta[g + f'(\tilde{k}_t)]}\right). \tag{10}$$

Therefore, in the case in which $\underline{c} < \tilde{c}_t < f(\tilde{k}_t)$, the optimal trajectory is provided by the following system of equations:

$$\begin{cases} \tilde{k}_{t+1} = g\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t \\ \tilde{c}_t = \dfrac{1}{\lambda} W\left(\dfrac{U'(\lambda \tilde{c}_{t-1})}{\delta[g + f'(\tilde{k}_t)]}\right). \end{cases} \tag{11}$$

One can prove [5], that system (11) has a stationary point (k^*, c^*) which is a saddle-point, and the behaviour of all the optimal trajectories is determined by the two singular trajectories which pass through the stationary point.

Applying the *Brock-Scheinkman* theorem [16], one can prove that the solution of system [11) is *globally asimptotically stable.*

An interesting case for the type of model studied is obtained when the utility function is of the Bernoulli-type:

$$U(x) = \frac{1}{1 - \nu} x^{1-\nu}. \tag{12}$$

In this case, system (11) becomes:

$$\begin{cases} \tilde{k}_{t+1} = g\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t \\ \tilde{c}_t = [\delta(g + f'(\tilde{k}_t))]^{1/\nu} \tilde{c}_{t-1} \end{cases} \tag{13}$$

and the co-ordinates of the singular point are:

$$\begin{cases} k^* = (f')^{-1}\left(\frac{1}{\delta} - g\right) \\ c^* = f(k^*) - (1-g)k^*. \end{cases}$$

In fig. 1 are represented the stationary trajectories:

$$(AB) : c = f(k) - (1-g)k \quad (\tilde{k}_{t+1} = \tilde{k}_t)$$
$$(CD) : k = k^* \quad (\tilde{c}_{t+1} = \tilde{c}_t)$$

as well as the curves:

$$c = f(k)$$
$$c = \underline{c}.$$

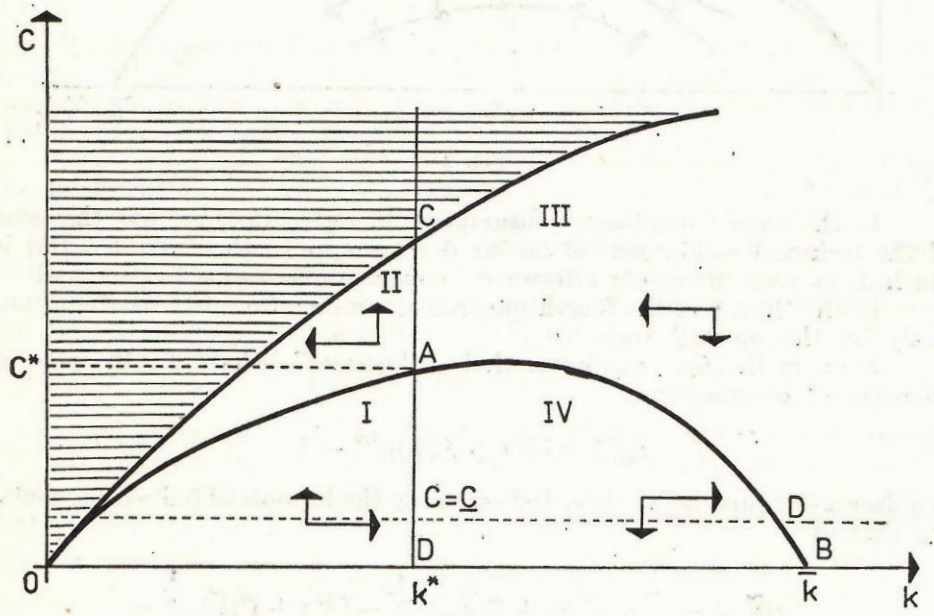


Fig. 1

The four curves determine in the (c, k) — plane four quadrants; in fig. 1 the directions of the optimal trajectories in every quadrant are also shown.

In fig. 2 are represented the singular trajectories which pass through the singular point (k^*, c^*), as well as the behaviour of the optimal trajectories in every of the four quadrants.

As we can see in fig. 2, the optimal trajectories have a "normal behaviour" only in the first quadrant, in the sense that the level of the technical endowment of labour k_t, as well as that of consumption c_t increase in time.

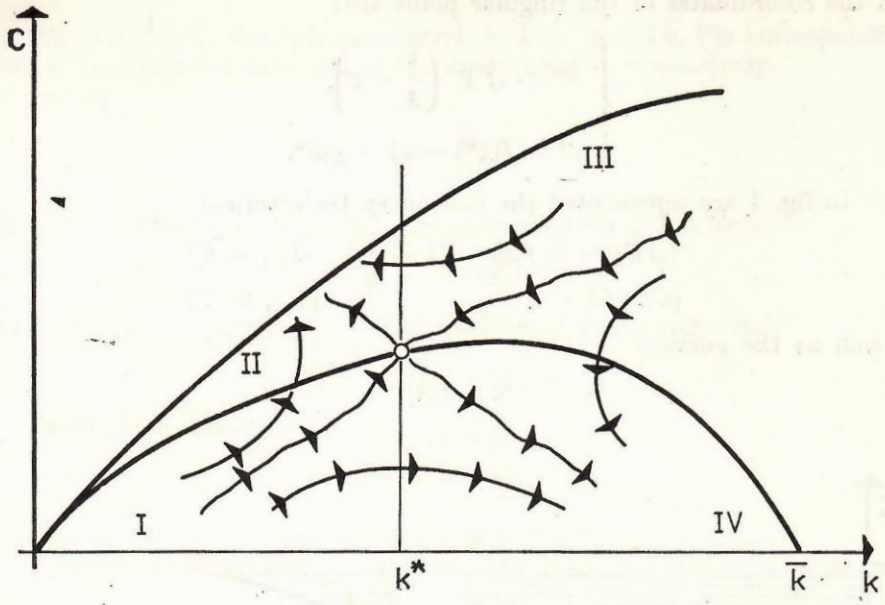


Fig. 2

In the second quadrant, consumption increases in time, but the level of the technical endowment of labour decreases, until the curve $c = f(k)$ is reached, at some moment; afterwards, consumption begins to decrease.

In the third and the fourth quadrants, consumption decreases continuously on the optimal trajectory.

Even in the first quadrant, that of "normal" behaviour, the rate of increase of consumption:

$$r(\tilde{k}_t) = [\delta(g + f'(k_t))]^{1/\nu} - 1$$

is a decreasing function of time. Indeed, using the formula of finite increments, we obtain:

$$r(\tilde{k}_{t+1}) - r(\tilde{k}_t) = [\delta(g + f'(\tilde{k}_{t+1}))]^{\frac{1}{\nu}} - [\delta(g + f'(\tilde{k}_t))]^{\frac{1}{\nu}} =$$

$$= \frac{1}{\nu}[\delta(g + f'(\xi))]^{\frac{1}{\nu}-1} \cdot f''(\xi)[\tilde{k}_{t+1} - \tilde{k}_t] < 0$$

$$\tilde{k}_t < \xi < \tilde{k}_{t+1}$$

since $\tilde{k}_{t+1} > \tilde{k}_t$ and $f''(\xi) < 0$.

We have to mention that the behaviour of the optimal trajectories is analogous in the general case, when we have an arbitrary utility function $U(.)$. This conclusion can be reached by analyzing system (11).

This behaviour of the optimal trajectory, characterized by increasing consumption, but with decreasing average rate, can contradict the dynamics of social needs, regarded as aspirations.

Further, the economic growth model will be completed by introducing an equation which will describe the dynamics of social needs or aspirations, respectively; the behaviour of the new model will be studied.

2. AN ECONOMIC GROWTH MODEL WHICH TAKES INTO ACCOUNT THE DYNAMICS OF SOCIAL NEEDS

Obviously, the possibilities of reflecting the mechanism of formation of social needs in an aggregative economic growth model are rather limited.

In what follows, we shall note by z_t the social need, regarded as aspiration, measured in value units, commensurable with those of consumption.

In a first approximation, the dynamics of social needs could be described by the equation:

$$z_{t+1} = \varphi_t(z_t,\ c_t,\ f(k_t)). \tag{14}$$

We started above from the assumption that social need (the aspiration) in year t+1 depends on its level in year t, on the actually reached level of consumption in year t, which, in its turn, generates new needs, and on the volume of production.

It is well known that the development, of production strongly influences the dynamics of social needs "... stirring up in the consumer the need the object of which is the object created by it" (K. Marx).

At the same time, the development of production, causing the increase of the civilization standard of society, generates new needs.

As function φ which models the dynamics of social need we could consider, for instance, a function of the form:

$$\varphi(c_t,\ z_t,\ f(k_t)) = (1 + r)c_t + \varepsilon_1(z_t - c_t) + \varepsilon_2 f(k_t). \tag{15}$$

Coefficients r, ε_1, ε_2 can be determined on the basis of sociological research.

As concerns the preformance of the model, we shall set ourselves the task of minimizing the square of the deviation between the social need, regarded as aspiration, and the volume of consumption actually allotted:

$$J_T = min\ \frac{1}{2}\sum_{t=0}^{T-1}[z_t - c_t]^2. \tag{16}$$

Obviously, we can take into account a discount factor, so that the performance be of the form:

$$J_T = min\ \frac{1}{2}\sum_{t=0}^{T-1}\delta^t[z_t - c_t]^2. \tag{16'}$$

We can consider also a more general case, in which the performance be of the form:

$$J_T = max\sum_{t=0}^{T-1}\delta^t\ U(c_t,\ z_t). \tag{17}$$

A model in which a performance of the form (16) has been considered was worked out and analyzed by N. E. Ryder Jr. and G. M. Heal [14].

In the Ryder Jr. — Heal model, the dynamic equation of the variable z is the following:

$$\dot{z} = \rho(c - z).$$

In the interpretation of the above mentioned authors, $z(t)$ defines the "stock of experience" in the domain of consumption:

$$z(t) = \rho e^{-\rho t} \cdot \int_{-\infty}^{t} e^{\rho s} c(s)\, ds$$

which is given as a "weighted average" of the past consumption, the weight decreasing exponentially as one moves away in the past.

In what follows, the optimality conditions and the qualitative properties of the following model will be analyzed:

$$\text{(II)} \quad \begin{cases} k_{t+1} = gk_t + f(k_t) - c_t; \quad k_0 = \alpha; \quad k_T \geqslant \omega \\ z_{t+1} = (1 + r)\, c_t + \varepsilon_1(z_t - c_t) + \varepsilon_2 f(k_t); \quad z_0 = \beta \\ J_T = \min\limits_{\underline{c} \leqslant c_t \leqslant f(k_t)} \dfrac{1}{2} \sum\limits_{t=0}^{T-1} [z_t - c_t]^2 \\ \underline{c} \leqslant c_t \leqslant f(k_t). \end{cases}$$

One can remark the absence of any discount factor in the expression of the performance; instead of it, a "penality" factor $\varepsilon_1(z_t - c_t)$ appears in the dynamic equation of the variable z_t.

4. THE QUALITATIVE STUDY OF THE OPTIMAL TRAJECTORIES

In order to infer the necessary optimality conditions, we shall use the Maximum principle for discrete systems.

The Hamiltonian function of the system is:

$$H_t(k_t, z_t, c_t, \psi_t^1, \psi_t^2) = \psi_t^1[gk_t + f(k_t) - c_t] +$$

$$+ \psi_t^2[(1 + r)c_t + \varepsilon_1(z_t - c_t) + \varepsilon_2 f(k_t)] - \frac{1}{2}[z_t - c_t]^2 + \qquad (18)$$

$$+ \lambda_t^1[f(k_t) - c_t] + \lambda_t^2[c_t - \underline{c}]$$

where the variables ψ_t^1 and ψ_t^2 are the solutions of the adjoint system:

$$\begin{cases} \psi_{t-1}^1 = \psi_t^1[g + f'(k_t)] + \psi_t^2 \varepsilon_2 f'(k_t) + \lambda_t^1 \cdot f'(k_t) \\ \psi_{t-1}^2 = \psi_t^2 \varepsilon_1 - [z_t - c_t]. \end{cases} \qquad (19)$$

Noting by $\tilde{c}_t$ the optimal control and by $\tilde{k}_t$, $\tilde{z}_t$, $\tilde{\psi}_t^1$, $\tilde{\psi}_t^2$ the corresponding values of the state variables and dual variables, respectively, we obtain the following *necessary optimality conditions* provided by the Maximum principle:

$$[-\tilde{\psi}_t^1 + (1 + r - \varepsilon_1)\tilde{\psi}_t^2 + [\tilde{z}_t - \tilde{c}_t] - \lambda_t^1 + \lambda_t^2][c_t - \tilde{c}_t] \leqslant 0 \tag{20}$$

for any feasible control $c_t \in [\underline{c}, f(k_t)]$ and for $t = 0, ..., T-1$;

$$\tilde{\psi}_t^1 \leqslant 0; \quad \tilde{\psi}_t^2 \leqslant 0; \quad \lambda_t^1 \geqslant 0; \quad \lambda_t^2 \geqslant 0 \tag{21}$$

for $t = 0, 1, ..., T-1$.

Moreover, the following conditions are satisfied:

$$\begin{cases} \lambda_t^1[f(\tilde{k}_t) - \tilde{c}_t] = 0 \\ \qquad\qquad\qquad\qquad t = 0, 1, ..., T-1 \\ \lambda_t^2[\tilde{c}_t - \underline{c}] = 0 \end{cases} \tag{22}$$

as well as the transversality condition:

$$\tilde{\psi}_{T-1}^1[\tilde{k}_T - \omega] = 0. \tag{23}$$

Theorem 1

The necessary optimality conditions provided by the Maximum principle are also sufficient optimality conditions and the optimal trajectory is unique.

We assumed above that

$$\bar{k}_T > \omega$$

where $\bar{k}_T$ is the solution of the equation

$$\bar{k}_{t+1} = g\bar{k}_t + f(\bar{k}_t) - \underline{c}; \quad \bar{k}_0 = \alpha.$$

From the analysis of conditions (20) it follows that the following situations can occur:

Phase I $\tilde{c}_t = \underline{c}$. This means that $\lambda_t^1 = 0$, $\lambda_t^2 \geqslant 0$;

the optimality condition (20) becomes:

$$[\tilde{z}_t - \underline{c}] - \tilde{\psi}_t^1 + (1 + r - \varepsilon_1)\tilde{\psi}_t^2 + \lambda_t^2 \leqslant 0 \tag{20'}$$

and the adjoint system is:

$$\begin{aligned} \tilde{\psi}_{t-1}^1 &= \tilde{\psi}_t^1[g + f'(\tilde{k}_t)] + \tilde{\psi}_t^2 \varepsilon_2 f'(\tilde{k}_t) \\ \tilde{\psi}_{t-1}^2 &= \varepsilon_1 \tilde{\psi}_t^2 - [\tilde{z}_t - \underline{c}]. \end{aligned} \tag{19'}$$

Phase II

(the "normal" case) $\underline{c} < \tilde{c}_t < f(\tilde{k}_t)$;

$$\lambda_t^1 = \lambda_t^2 = 0$$

$$\tilde{z}_t - \tilde{c}_t = \tilde{\psi}_t^1 - (1 + r - \varepsilon_1)\,\tilde{\psi}_t^2 \tag{20''}$$

$$\begin{cases} \tilde{\psi}_{t-1}^1 = \tilde{\psi}_t^1[g + f'(\tilde{k}_t)] + \tilde{\psi}_t^2 \varepsilon_2 f'(\tilde{k}_t) \\ \tilde{\psi}_{t-1}^2 = \varepsilon_1 \tilde{\psi}_t^2 - [\tilde{z}_t - \tilde{c}_t]. \end{cases} \tag{19''}$$

Phase III. $\tilde{c}_t = f(\tilde{k}_t)$ and hence $\lambda_t^2 = 0$, $\lambda_t^1 \geqslant 0$.

The optimality condition and the adjoint system become, respectively:

$$[\tilde{z}_t - f(\tilde{k}_t)] - \tilde{\psi}_t^1 + (1 + r - \varepsilon_1)\,\tilde{\psi}_t^2 - \lambda_t^1 \geqslant 0, \tag{20'''}$$

$$\begin{cases} \tilde{\psi}_{t-1}^1 = \tilde{\psi}_t^1[g + f'(\tilde{k}_t)] + [\tilde{\psi}_t^2 \varepsilon_2 + \lambda_t^1] f'(\tilde{k}_t) \\ \tilde{\psi}_{t-1}^2 = \varepsilon_1 \tilde{\psi}_t^2 - [\tilde{z}_t - f(\tilde{k}_t)]. \end{cases} \tag{19'''}$$

Obviously, the optimal trajectory can contain one, two or all tbe three phases.

In order to perform the qualitative study of the model, we shall identify the singular points and we shall examine the local behaviour of the trajectories around these points.

Taking into account the dynamic equation of the variable k_t

$$k_{t+1} = g k_t + f(k_t) - c_t$$

we obtain the stationary value of this variable as a solution of the equation:

$$c_t = f(k_t) - (1 - g)\,k_t. \tag{24}$$

From equation (24) it follows that, for the trajectories corresponding to phase III, there exist no singular points.

For the "normal" case (phase II), using the optimality condition (20 II), we can write the *Hamiltonian system*:

$$\begin{cases} \tilde{k}_{t+1} = g\tilde{k}_t + f(\tilde{k}_t) - \tilde{z}_t + \tilde{\psi}_t^1 - (1 + r - \varepsilon_1)\,\tilde{\psi}_t^2 \\ \tilde{z}_{t+1} = \varepsilon_2 f(\tilde{k}_t) + (1 + r)\,\tilde{z}_t - (1 + r - \varepsilon_1)\,\tilde{\psi}_t^1 + (1 + r - \varepsilon_1)^2 \tilde{\psi}_t^2 \\ \tilde{\psi}_{t-1}^1 = [g + f'(\tilde{k}_t)]\,\tilde{\psi}_t^1 + \varepsilon_2 f'(\tilde{k}_t)\,\tilde{\psi}_t^2 \\ \tilde{\psi}_{t-1}^2 = -\tilde{\psi}_t^1 + (1 + r)\,\tilde{\psi}_t^2. \end{cases} \tag{25}$$

Noting

$$\tilde{H}_t^2(\tilde{k}_t, \tilde{z}_t, \tilde{\psi}_t^1, \tilde{\psi}_t^2) = \max_{\underline{c} < c_t < f(k_t)} H(k_t, z_t, c_t, \psi_t^1\, \psi_t^2) \tag{26}$$

we have:

$$\widetilde{H}_t^2(\widetilde{k}_t,\ \widetilde{z}_t\ \widetilde{\psi}_t^1,\ \widetilde{\psi}_t^2) = \widetilde{\psi}_t^1[g\widetilde{k}_t + f(\widetilde{k}_t)] + \widetilde{\psi}_t^2[(1+r)\,z_t + \varepsilon_2 f(k_t)] - z_t\psi_t^1 +$$

$$+ \frac{1}{2}(\widetilde{\psi}_t^1)^2 + \frac{1}{2}(1+r-\varepsilon_1)^2\,\widetilde{\psi}_t^2 - (1+r-\varepsilon_1)\,\widetilde{\psi}_t^1\widetilde{\psi}_t^2. \tag{27}$$

In this case, the Hamiltonian system (25) can be written as:

$$\begin{cases} \widetilde{k}_{t+1} = \dfrac{\partial \widetilde{H}_t^2}{\partial \psi_t^1} \\ \widetilde{z}_{t+1} = \dfrac{\partial \widetilde{H}_t^2}{\partial \psi_t^2} \\ \widetilde{\psi}_{t-1}^1 = \dfrac{\partial \widetilde{H}_t^2}{\partial k_t} \\ \widetilde{\psi}_{t-1}^2 = \dfrac{\partial \widetilde{H}_t^2}{\partial z_t}. \end{cases} \tag{28}$$

Noting:

$$\widetilde{y}_t^i = \widetilde{\psi}_{t-1}^i; \quad i = 1,\ 2 \tag{29}$$

system (25) becomes:

$$\begin{cases} \widetilde{k}_{t+1} = g\widetilde{k}_t + f(\widetilde{k}_t) - \widetilde{z}_t + \widetilde{y}_{t+1}^1 - (1+r-\varepsilon_1)\,y_{t+1}^2 \\ \widetilde{z}_{t+1} = \varepsilon_2 f(\widetilde{k}_t) + (1+r)\,\widetilde{z}_t - (1+r-\varepsilon_1)\,\widetilde{y}_{t+1}^1 + (1+r-\varepsilon_1)^2\,\widetilde{y}_{t+1}^2 \\ \widetilde{y}_t^1 = [g + f'(\widetilde{k}_t)]\,\widetilde{y}_{t+1}^1 + \varepsilon_2 f'(\widetilde{k}_t)\widetilde{y}_{t+1}^2 \\ \widetilde{y}_t^2 = -\widetilde{y}_{t+1}^1 + (1+r)\,\widetilde{y}_{t+1}^2 \end{cases} \tag{30}$$

which can be written also as:

$$\begin{cases} \widetilde{k}_{t+1} - \widetilde{y}_{t+1}^1 + (1+r-\varepsilon_1)\,\widetilde{y}_{t+1}^2 = g\widetilde{k}_t + f(\widetilde{k}_t) - \widetilde{z}_t \\ \widetilde{z}_{t+1} + (1+r-\varepsilon_1)\,\widetilde{y}_{t+1}^1 - (1+r-\varepsilon_1)^2\,\widetilde{y}_{t+1}^2 = \varepsilon_2 f(\widetilde{k}_t) + (1+r)\,\widetilde{z}_t \\ [g + f'(\widetilde{k}_t)]\,\widetilde{y}_{t+1}^1 + \varepsilon_2 f'(\widetilde{k}_t)\widetilde{y}_{t+1}^2 = \widetilde{y}_t^1 \\ -\widetilde{y}_{t+1}^1 + (1+r)\,\widetilde{y}_{t+1}^2 = \widetilde{y}_t^2. \end{cases} \tag{31}$$

We note

$$\begin{cases} A(k_t) = \begin{pmatrix} gk_t + f(k_t) & \varepsilon_2 f(k_t) \\ 0 & 0 \end{pmatrix} \\ B = \begin{pmatrix} 0 & 0 \\ -1 & 1+r \end{pmatrix} \\ C = \begin{pmatrix} -1 & 1+r-\varepsilon_1 \\ 1+r-\varepsilon_1 & -(1+r-\varepsilon_1)^2 \end{pmatrix} \\ x_t = \begin{pmatrix} k_t \\ z_t \end{pmatrix}; \quad y_t = \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix}. \end{cases} \tag{32}$$

System (31) can be written in the following way:

$$\begin{bmatrix} E & C \\ 0 & A'(k_t) + B \end{bmatrix} \cdot \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} A^T(\tilde{k}_t) + B & 0 \\ 0 & E \end{bmatrix} \cdot \begin{bmatrix} x_t \\ y_t \end{bmatrix}. \tag{33}$$

Since the matrix $A'(k_t) + B$ is nonsingulary, system (33) can be transformed as follows:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} E & -C[A'(k_t) + B]^{-1} \\ 0 & [A'(k_t) + B]^{-1} \end{bmatrix} \cdot \begin{bmatrix} [A(\tilde{k}_t) + B]^* & 0 \\ 0 & E \end{bmatrix} \cdot \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \tag{34}$$

or

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \\ \tilde{y}_{t+1}^1 \\ \tilde{y}_{t+1}^2 \end{bmatrix} = \begin{bmatrix} [A(\tilde{k}_t) + B]^* & -C[A'(\tilde{k}_t) + B]^{-1} \\ 0 & [A'(\tilde{k}_t) + B]^{-1} \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \tilde{y}_t^1 \\ \tilde{y}_t^2 \end{bmatrix}. \tag{35}$$

Setting the conditions:

$$\begin{aligned} &\tilde{k}_{t+1} = \tilde{k}_t = \hat{k}; \quad \tilde{z}_{t+1} = \tilde{z}_t = \hat{z} \\ &\tilde{y}_{t+1}^1 = \tilde{y}_t^1 = \hat{y}^1 = \hat{\psi}^1; \quad \tilde{y}_{t+1}^2 = \tilde{y}_t^2 = \hat{y}^2 = \hat{\psi}^2 \end{aligned} \tag{36}$$

we obtain from (35) the co-ordinates of the *singular point* of the Hamiltonian system (25):

$$\begin{cases} \hat{z} = \left(1 + \dfrac{r+\varepsilon_2}{1-\varepsilon_1}\right) f(\hat{k}) - \left(1 + \dfrac{r}{1-\varepsilon_1}\right)(1-g)\,\hat{k} \\ \hat{y}^2 = -\dfrac{r+\varepsilon_2}{(1-\varepsilon_1)^2} f(\hat{k}) + \dfrac{r}{(1-\varepsilon_1)^2}(1-g)\hat{k} \\ \hat{y}^1 = r\hat{y}^2 \end{cases} \tag{37}$$

where $\hat{k}$ is the solution of equation

$$\det [A'(k) + B] = 1 \tag{38}$$

which assures that the matrix of system (35) had $\lambda = 1$ as eigenvalue.

From (38) we obtain:

$$g + \left(1 + \frac{\varepsilon_2}{r}\right) f'(\hat{k}) = 1 \tag{39}$$

which represents the "modified golden rule" for model (II).

Noting by k^* the solution of equation

$$g + f'(k) = 1,$$

which represents the ordinary "golden rule", we notice that $\hat{k} \leqslant k^*$.

We have $\hat{k} = k^*$ only in the case in which $\varepsilon_2 = 0$.

In order to examine the behaviour of the optimal trajectory in the neighbourhood of the singular point, we note:

$$\begin{cases} \xi_t^1 = \tilde{k}_t - \hat{k}\,; \quad \xi_t^2 = \tilde{z}_t - \hat{z} \\ \eta_t^1 = \tilde{y}_t^1 - \hat{y}^1\,; \quad \eta_t^2 = \tilde{y}_t^2 - \hat{y}^2. \end{cases} \tag{40}$$

Developing the functions in system (35) in Taylor-series in the neighbourhood of the singular point, we obtain:

$$\begin{bmatrix} \xi_{t+1}^1 \\ \xi_{t+1}^2 \\ \eta_{t+1}^1 \\ \eta_{t+1}^2 \end{bmatrix} = M \begin{bmatrix} \xi_t^1 \\ \xi_t^2 \\ \eta_t^1 \\ \eta_t^2 \end{bmatrix} + R(\cdot) \tag{41}$$

where: (42)

$$M = \begin{bmatrix} b - \varepsilon_1 d & -1 & \frac{\varepsilon_1}{a} & -1 + \frac{\varepsilon_1}{a} \cdot b \\ (1-b)r + (1+r-\varepsilon_1)\varepsilon_1 d & 1+r-\lambda & -(1+r-\varepsilon_1)\frac{\varepsilon_1}{a} & (1+r-\varepsilon_1)\left(1 - \frac{\varepsilon_1}{a} \cdot b\right) \\ -(1+r)d & 0 & \frac{1+r}{a} & -\frac{(1+b)r}{a} \\ -d & 0 & \frac{1}{a} & \frac{b}{a} - \lambda \end{bmatrix}$$

The following notations have been used:

$$\begin{cases} a = [(1+r)f'(\hat{k}) + g] + \varepsilon_2 f'(\hat{k}) = f'(\hat{k}) + g + r \\ b = f'(\hat{k}) + g = a - r \\ d = \dfrac{r+\varepsilon_2}{a}\hat{y}^2 f''(\hat{k}). \end{cases} \tag{43}$$

The characteristic equation of system (41) is:

$$\det [M - \lambda E] = 0. \tag{44}$$

We remark that, if λ is *an eigenvalue of matrix M, then* $\dfrac{1}{\lambda}$ *is also an eigenvalue of this matrix.*

This property is analogous to that proved by M. Kurz for the Hamiltonian systems with continuous variables, where the matrix of the system admits, together with the eigenvalue λ, the eigenvalue $-\lambda$.

From (44) we obtain:

$$\begin{cases} \lambda^4 - \left(2 + a + \dfrac{1}{a} - \varepsilon_1 d\right)\lambda^3 + 2\left(1 + a + \dfrac{1}{a} - \dfrac{1+\varepsilon_1}{2}d\right)\lambda^2 - \\ \qquad - \left(2 + a + \dfrac{1}{a} - \varepsilon_1 d\right)\lambda + 1 = 0. \end{cases} \tag{45}$$

As we could expect, equation (45) is reciprocal.

Noting

$$x = \lambda + \frac{1}{\lambda}. \tag{46}$$

We obtain:

$$x^2 - \left(2 + a + \frac{1}{a} - \varepsilon_1 d\right)x + 2\left(a + \frac{1}{a} - \frac{1+\varepsilon_1}{2}d\right) = 0. \tag{47}$$

Equation (47) has real roots. Indeed,

$$\begin{aligned} \Delta x &= \left[2 + a + \frac{1}{a} - \varepsilon_1 d\right]^2 - 8\left(a + \frac{1}{a} - \frac{1+\varepsilon_1}{2}d\right) = \\ &= \left[-2 + a + \frac{1}{a} - \varepsilon_1 d\right]^2 > 0. \end{aligned} \tag{48}$$

Noting trinomial (47) by $h(x)$, we have:

$$h(2) = -(1 - \varepsilon_1)d < 0 \tag{49}$$

since $\varepsilon_1 < 1$ and $d > 0$, and

$$h(-2) = 8 + 4\left(a + \frac{1}{a}\right) - (3 + \varepsilon_1)d. \tag{50}$$

Taking into account the expression of d given by relation (43), we may assume that $h(-2) < 0$ and hence, the roots of equation (47) are in the following situation:

$$x_1 > 2; \qquad |x_2| < 2. \tag{51}$$

The characteristic roots of system (41) are:

$$\lambda_{1,2,3,4} = \frac{x_{1,2} \pm \sqrt{x_{1,2}^2 - 4}}{2} \tag{52}$$

where $x_{1,2}$ are the roots of equation (47).

Since $|x_2| < 2$, it follows that

$$|\lambda_{3,4}|^2 = \frac{x_2^2 + |x_2^2 - 4|}{4} < 1.$$

The above results allow the following conclusion:

Proposition 1

The linear approximation of the Hamiltonian system (41) has two real eigenvalues, one of which is stable and the other unstable:

$$\lambda_1 < 1; \quad \lambda_2 = \frac{1}{\lambda_1} > 1 \tag{53}$$

and two stable complex eigenvalues:

$$\lambda_3 = \frac{1}{\lambda_4}$$

with

$$|\lambda_{3,4}| < 1.$$

We pass now to the identification of the properties of the optimal trajectories in various zones of the positive octant of the space (k, z, ψ^1, ψ^2) delimited by the stationary surfaces:

$$\begin{cases} \tilde{k}_{t+1} = \tilde{k}_t = k; \quad \tilde{z}_{t+1} = \tilde{z}_t = z \\ \tilde{\psi}_{t-1}^1 = \tilde{\psi}_t^1 = \psi^1; \quad \tilde{\psi}_{t-1}^2 = \tilde{\psi}_t^2 = \psi^2. \end{cases} \tag{54}$$

To begin with, we notice that, taking into account the last equation of (25), the optimality condition (20 II) can be written:

$$\tilde{z}_t - \tilde{c}_t = \varepsilon_1 \tilde{\psi}_t^2 - \tilde{\psi}_{t-1}^2 \tag{55}$$

and condition $\tilde{\psi}^2_{t-1} = \tilde{\psi}^2_t$ implies:

$$\tilde{\psi}^1_t = r\psi^2_t \tag{56}$$

We note also:

$$\varphi(k) = f(k) - (1 - g)k \tag{57}$$

and we notice that, for $k \leqslant k^*$, function $\varphi(k)$ is increasing; k^* (the golden rule) is the solution of equation:

$$f'(k) + g = 1. \tag{58}$$

With these notations, formulas (37), which give the coordinates of the singular point can be written in the following way:

$$\begin{cases} \hat{z} = \left(1 + \dfrac{r}{1 - \varepsilon_1}\right) \varphi(\hat{k}) + \dfrac{\varepsilon_2}{1 - \varepsilon_1} \hat{k} \\ \hat{\psi}^2 = -\dfrac{r}{(1 - \varepsilon_1)^2} \varphi(\hat{k}) - \dfrac{\varepsilon_2}{(1 - \varepsilon_1)^2} \hat{k} \\ \hat{\psi}^1 = r\hat{\psi}^2 \end{cases} \tag{59}$$

where $\hat{k}$ is the solution of equation:

$$\varphi'(k) + \frac{\varepsilon_2}{r} f'(k) = 1. \tag{60}$$

The corresponding volume of consumption is

$$\hat{c} = \varphi(\hat{k}). \tag{61}$$

Proposition 2

Let us consider:

$$\varphi(k) = f(k) - (1 - g)k \tag{62}$$

and

$$h(k_t, c_t) = \frac{1 + r - \varepsilon_1}{1 - \varepsilon_1} c_t + \frac{\varepsilon_2}{1 - \varepsilon_1} f(k_t). \tag{63}$$

We have the following monotonicity properties:

$$[\tilde{k}_{t+1} - \tilde{k}_t][\varphi(\tilde{k}_t) - \tilde{c}_t] \geqslant 0 \tag{64}$$

$$[\tilde{z}_{t+1} - \tilde{z}_t][h(\tilde{k}_t, \tilde{c}_t) - \tilde{z}_t] \geqslant 0. \tag{65}$$

Proof: The dynamic equation of the variable $\tilde{k}$:

$$\tilde{k}_{t+1} = g\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t$$

can be also written as:

$$\tilde{k}_{t+1} - \tilde{k}_t = \varphi(\tilde{k}_t) - \tilde{c}_t.$$

This relation proves (64).

The dynamic equation of $\tilde{z}_t$ is:

$$\tilde{z}_{t+1} = (1 + r - \varepsilon_1)\tilde{c}_t + \varepsilon_1 \tilde{z}_t + \varepsilon_2 f(\tilde{k}_t)$$

wherefrom:

$$\tilde{z}_{t+1} - \tilde{z}_t = (1 - \varepsilon_1)\left[\frac{1 + r - \varepsilon_1}{1 - \varepsilon_1}\tilde{c}_t + \frac{\varepsilon_2}{1 - \varepsilon_1} f(\tilde{k}_t) - \tilde{z}_t\right]$$

or

$$\tilde{z}_{t+1} - \tilde{z}_t = (1 - \varepsilon_1)[h(\tilde{k}, \tilde{c}_t) - \tilde{z}_t].$$

This proves relation (65).

Proposition 3

Let us consider

$$\Delta_t = \tilde{z}_t - \tilde{c}_t. \tag{66}$$

the deviation of the social need from consumption and

$$\delta_t = \tilde{\psi}_t^1 - r\tilde{\psi}_t^2. \tag{67}$$

Then, we have the following property:

$$\Delta_{t+1} - \Delta_t = \varepsilon_1 \delta_{t+1} - \delta_t.$$

Proof: From (51) it follows:

$$\Delta_{t+1} - \Delta_t = \varepsilon_1[\tilde{\psi}_{t+1}^2 - \tilde{\psi}_t^2] - [\tilde{\psi}_t^2 - \tilde{\psi}_{t-1}^2] = \varepsilon_1[\tilde{\psi}_{t+1}^1 - r\tilde{\psi}_{t+1}^2] - [\tilde{\psi}_t^1 - r\tilde{\psi}_t^2]$$

or

$$\Delta_{t+1} - \Delta_t = \varepsilon_1 \delta_{t+1} - \delta_t.$$

The proved properties provide information concerning the behaviour of the optimal trajectories of model II, in which we have in view the minimization of the square of the deviation between the social need and the volume of the allotted consumption.

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LA DYNAMIQUE DES BESOINS SOCIAUX ET LA CROISSANCE ÉCONOMIQUE

Dans cet article on justifie la necessité d'inclure dans les modèles de croissance économique les éléments qui comportent la description de la dynamique des besoins sociaux. Ensuite on propose un modèle dynamique de croissance économique qui prenne en considération la dynamique des besoins sociaux. Pour le modèle proposé on deduit les conditions optimales et l'on demontre certaines propriétés qualitatives des trajectoires optimales.

ДИНАМИКА ОБЩЕСТВЕННЫХ ПОТРЕБНОСТЕЙ И ЭКОНОМИЧЕСКИЙ РОСТ

В статье обосновывается необходимость включения в модели экономического роста элементов, описывающих динамику общественных потребностей. Далее в работе предлогается динамическая модель экономического роста, в которои учитывается динамика общественных потребностей. Для предлогаемои модели выведены условия оптимума и доказываются некоторые качественные свойства оптимальных траекторий.

15. 02. 1979.