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On the Job Shop Scheduling by Mathematical Programming

The authors have introduced in [6] *the function as a continuous alter-*

$$\sigma(x, \alpha) = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} [p_j x_j/(\alpha_j + x_j)], \quad x \geqslant 0, \ \alpha > 0, \ p \geqslant 0$$

native to the mixed variables objective-function for optimizing the activity of production and allocation analysis taking into account the set-up costs. We build in this paper several models having at their basis this type of objective-function for the job shop scheduling wherein we suppose that the major problems constraints can be linearly modelled.

1. A CONTINUOUS FUNCTION WITH ALMOST EVERYWHERE-BOOLEAN VALUES

There are many problems wherein, at the same time with "the costs", depending on the extent of an activity, there intervenes also a cost — or a bonus — connected to the fact that the activity is at zero level or greater than zero. Usually, this situation is modelled through introducing at the same time with variable x_j, a new variable δ_j, having boolean values, which is equal to 0 if x_j is zero, and equal to 1 if $x_j > 0$. The introduction of these variables generally models the process as a mixed variables model — continous and integer — and increases the number of variables. We said "generally" because there are also examples where one succeeds, taking into account the properties of the optimal solution, to let the process be modelled only through continuous variables.

In [6] the authors propose that the "set-up costs" of the activities to be added by means of functions

$$\delta_j(x_j, \alpha_j) = x_j / (\alpha_j + x_j), \quad \alpha_j > 0$$

associated to activities. As $\delta_j(0, \alpha_j) = 0$ and $\delta_j(x_j, \alpha_j) = 1 - \alpha_j / (\alpha_j + x_j)$ the function δ fulfils its proposed purpose, if x_j is zero or big enough as against α_j. For instance, the solution of problem

$$min\ \sigma(x, 1) = \sum_{j=1}^{n} [x_j/(1 + x_j)],\ x \geqslant 0,\ x \in S \qquad \text{gives}$$

a minimal number of the positive components of $x \in S$, x big enough as against 1. More precisely, we have [6] the following theorem:

There exists $\alpha_j > 0$ so that the solution of problem (σ)

$$min\ \sigma(x, \alpha, = \sum_{j=1}^{n} x_j/(\alpha_j + x_j),\ Ax = b,\ x \geqslant 0$$

is a solution of the cardinal minimization problem (CMP): $min\ |x|^+ Ax = b$, $x \geqslant 0$ where $|x|^+$ denotes the number of the strictly positive components o x, i.e. the cardinal of x.

2. SOME PROBLEMS RELATED TO ASSIGNMENT OF ITEMS TO MACHINES

2.1. Let us suppose a workshop production process wherein we denote by x_{ijk} the quantity of item i, $i = 1, ..., n$, processed in period j, $j = 1, ..., s$, on machine k, $k = 1, ..., m$. For the models known by the authors from practice [1], as for instance the production programming in nails, electrodes, welded wirenetting etc. workshops, the dimensions of the problem do not exceed $n \leqslant 100$, $s \leqslant 4$, $k \leqslant 25$, the constraints number being under 200.

The main constraints of problem P_1 are as follows:

— constraints concerning the covering of demands

$$\sum_{j=1}^{s} \sum_{k=1}^{m} x_{ijk} \geqslant d_i,\ i = 1, ..., n \tag{1}$$

— constraints concerning the capacity c_{jk} of each k machine in each time period j,

$$\sum_{i=1}^{n} t_{ik}\, x_{ijk} \leqslant c_{jk}, \qquad \begin{matrix} j = 1, ..., s \\ k = 1, ..., m \end{matrix} \tag{2}$$

— other constraints of some bottleneck, as for instance the nail "drum-cleaning" capacity, the "calcination" of the electrodes or the stocking of the wirenets capacity in each period:

$$\sum_{i=1}^{n} \sum_{k=1}^{m} \beta_{ik}\, x_{ijk} \leqslant h_j,\ j = 1, ..., s. \tag{3}$$

If we attach to conditions (1), (2), (3) of problem (P1) the following objective function, we obtain the model M_1:

$$min \sum_{i=1}^{n} \sum_{k=1}^{m} p_{ik} \frac{\sum_{j=1}^{s} x_{ijk}}{\alpha_{ik} + \sum_{j=1}^{s} x_{ijk}},\quad \alpha_{ik} > 0,\ p_{ik} \geqslant 0. \tag{4}$$

If all p_{ik} are equal, then the optimal solution minimizes the total number of startings of the product fabrication on all m machines. We can carry out a minimal number of products with a certain machine, if we select adequately coefficients p_{ik}. In this last case, we can also use the function:

$$min \sum_{i=1}^{n} p_{i\cdot} \left[\sum_{j=1}^{s} x_{ijk_0} / (\alpha_i + \sum_{j=1}^{s} x_{ijk_0}) \right], \quad p_{i\cdot} \geq 0, \quad \alpha_i > 0. \tag{4a}$$

If we relax the problem by not taking into account the periods and if we denote by $x_{i\cdot k}$ the amount of product i to be processed on machine k in all periods s, then the model M_2 of the problem — certainly much more approximate, as we can no longer be sure of its effective achievement at period level — can be written as follows:

$$\sum_{k=1}^{m} x_{i\cdot k} \geq d_{i\cdot} \quad i = 1, \ldots, n \tag{1'}$$

$$\sum_{i=1}^{n} t_{ik}\, x_{i\cdot k} \leq c_{k\cdot} \quad k = 1, \ldots, m \tag{2'}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m} \beta_{ik}\, x_{i\cdot k} \leq h. \tag{3'}$$

$$min \sum_{i=1}^{n} \sum_{k=1}^{m} p_{i\cdot k}\, x_{i.k} / (\alpha_{ik} + x_{i\cdot k}), \quad p_{ik} \geq 0 \quad \alpha_{ik} > 0 \tag{4'}$$

or analogously to (4a):

$$min \sum_{i=1}^{n} p_{i\cdot}\, x_{i.k_0} / (\alpha_i + x_{i.k_0}), \quad p_i \geq 0, \quad \alpha_i > 0. \tag{4'a}$$

2.2. *Problem of items assignment to machines with the minimizing of the production cost and of set-up cost.*

The problem of items assignment to machines can be considered also as a problem of minimizing the production cost and set-up cost. Such a problem, well known in the literature has as model:

$$min \sum_{j=1}^{n} f_j(x_j), \quad Ax \geq b, \quad x \geq 0$$

$$\text{where } f_j(x_j) = \gamma_j x_j + P_j \delta_j, \quad P_j \geq 0, \quad \delta_j = \begin{cases} 0 & \text{if } x_j = 0 \\ 1 & \text{if } x_j > 0 \end{cases}.$$

Using the function defined by us, this problem can new be modelled as:

$$min \sum_{j=1}^{n} f_j(x_j), \text{ where } f_j(x_j) = \gamma_j x_j + P_j \frac{x_j}{\alpha_j + x_j}, \quad P_j \geq 0 \tag{5}$$

subject to the constraints $Ax \geq b$, $x \geq 0$.

So we have model M_3:

$$min \sum_{i=1}^{n} \sum_{k=1}^{m} \gamma_{ik}\ x_{i \cdot k} + \sum_{i=1}^{n} \sum_{k=1}^{m} P_{ik}[x_{i \cdot k}/(\alpha_{ik} + x_{i \cdot k})] \tag{6}$$

with conditions (1'), (2') and (3') and where γ_{ik} represents the cost of processing a unit of product i on machine k and P_{ik} is the set-up cost.

2.3. *A model with constraints on the dispersion of the fabrication process*

Let us suppose that in order to fulfil the demand d_i of item i, we want to have a production process, wherein only at the utmost m_i machines would take part. If V_k is a subset of K_i, $V_k \subset K_i$, the set K_i of the machines able to process item i, then this condition can be imposed as constraint of the form:

$$\sum_{k \in V_k} x_{i \cdot k}/(\alpha_{ik} + x_{i \cdot k}) \leqslant m_i. \tag{7}$$

Similarly, we can write a constraint which imposes that, on a certain machine k, at the most n_k items may be processed. | If the items are in a certain subset V_i of item i belonging to the set I_k of the items which can be processed on machine k, we have:

$$\sum_{i \in V_i} x_{i \cdot k}/(\alpha_{ik} + x_{i \cdot k}) \leqslant n_k. \tag{8}$$

Let us denote M4 the model having (6) as an objective-function, and as constraints (1'), (2'), (3') and in addition (7) and /or (8).

2.4. *Models with continuity constraints of the production process*

One of the most difficult problems in production programming is that of ensuring a continuous production process on machines as soon as the production of an item on a machine has been decided. We have not a model for the general problem, but we notice that a constraint of the form:

$$\prod_{j \in V_j} [x_{i_0 j k_0}/(\alpha_{i_0 j k_0} + x_{i_0 j k_0}) \geqslant 1 - \lambda \tag{9}$$

where $0 < \lambda < 1$ achieves, without specifying the level, a continuous production, at periods V_j, of item i_0 on the machine k_0. If need be, λ can be also considered as an unknown. A model called M5 with nonlinearities of type (7) (8), or (9) in constraints is considered more difficult than a model wherein the non-linearities appear in the objective function. So that we consider it preferable to try to solve a M_6-type model:

$$max \prod_{j \in V_j} x_{i_0 j k_0}/(\alpha_{i_0 j k_0} + x_{i_0 j k_0}) \tag{10}$$

with the linear constraints (1), (2), (3).

2.5. *A model with efficient loading conditions of machines*

For a certain production processes, we require that the loading of a machine with a certain item to be made either at a level closed to machine capacity or at zero level, such that the machine to remain free for processing another item. Such a condition may be written as:

$$x_{ijk} \geqslant \beta\, q_{ijk}[x_{ijk}/(\alpha_{i.k} + x_{ijk})] \tag{11}$$

where q_{ijk} is the maximal quantity of item i that may be obtained on machine k in period j and β is a given percentage.
So we can consider a model M7 defined by (1), (2), (3), (11) and (4) or by (1'), (2'), (3'), (4') and (12),

$$x_{i\cdot k} \geqslant q_{i\cdot k}[x_{i\cdot k}/(\alpha_{i\cdot k} + x_{i\cdot k})] \tag{12}$$

where q_{ik} is the maximal production obtained for item i on machine k during all the planning horizon.

3. ON THE SOLVABILITY OF MODELS AND RELATED RESEARCHES

The difficulty of solving some general problems of programming n activities on m machines is well known, due to their large dimensions, even when n or m of the order of tenth. The intelligent use of the various models can be really useful, either in obtaining a suboptimal solution or even in obtaining of an admissible solution .

From the mathematical statement of a problem and from the study of the resulted model, we can infer qualitative properties of the optimal solution and through this we can, at the same time, limit even more the set wherein the optimal solution is seeked after.

Thus, as a consequence of modelling the problem of assigning activities on machines under the form M_2 and M_3, we can state: "The optimal solution of an assignment problem with linear constraints , which minimize the total number of changeovers in all periods of all the m machines (Model M_2) or the optimal solution which minimizes the production cost and the set-up cost (Model M_3), occurs in a vertex". The validity of this statement is immediate, taking into account that the objective-functions of models M_2 and M_3 are concave and that it is a known fact that the minimum of a concave function over a bounded polyhedral set is obtained in a vertex. The bouding condition can be easily obtained through imposing some limits to the variables.

Though the above statement reduces the seeking for the solution from a set with the cardinality of the continuum to a finite set of vertexes, this set, for real problems, can be big enough for making a complete ennumeration procedure to be unattainable. This is why it is recommended to draw up some procedures which, taking into account the information provided by the objective functions, are able to analyse only part of the vertexes. Besides, the efficiency function is also separable, particularly for models M_2 and M_3.

The authors consider that an ultimate answer to the problem of solvability of some models can be given only after having effected numerical tests on real problems of various dimensions. Unless we shall be in possesion of such results, we are obliged to close here our comments, not without mentioning that the basis for drawing up such procedures can be found in [2], [7], [11], [17] [19]. The above stated results have been directly drawn up starting from [1] and [6].

The references covering the production programming are very large. In the following we shall briefly mention few of them concerning the above discussed matters.

One of the oldest models (1958) is Manne's model [15], a model which integer in the first stage, is finally transformed into a linear programming problem of large dimensions. Manne's technique is applied subsequently by many authors; for instance Gorenstain [10] applies it to a problem of tyre production programming.

In a process of assigning items on machines, it is required within a certain period, if a product was assigned to a machine, to carry out on the machine the greatest share of the lot to be produced. This is necessary in order to minimize, for instance, the cost — in time or through wear — of the production changeovers from a machine to another one [13].

We mention that this problem can not be modelled only like a mathematical programming model. Dorsey Hodgson and Ratliff state in [4], [5] this problem like a minimal flow problem, but without taking into account the costs of production changeovers. Love and Vemuganti [14] state this problem as a minimal flow problem, but they take into account those costs. From the mathematical view point the problem discussed here has at present new researching directions. These new directions got their own names: cardinality constrained linear programming [18], [20], lattice point linear programming [3], [9], [12], [18], integer programming and non-convex programming [8], [11], [16, etc.

Now, we mention among the results exposed in the quoted references, which are of a particular interest, that given by Cabot in [3] and which is similar with our result. Starting from a (GLP) problem of lattice-point linear programming, $min \sum_{j=1}^{n} c_j x_j = z$ with the usual condition $x=(x_1, ..., x_n) \in S$, $S = \left\{ x \,\middle|\, \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, ..., m,\ x_j \geqslant 0,\ j = 1, ..., n \right\}$, to which nevertheless the supplementary condition no more than r variables from a specified set $V_k = \{x_1, ..., x_k\}$ be strictly positive, Cabot considered a GLP_0 version of that problem:

Let be found, if there exists, an $x \in S \cap Q$ where $Q = \{(x_1, ..., x_n) |$ with at the most r components of subset V_k are strictly positive$\}$. Cabot, introducing the function $y = \sum_{x_j \in V_k} 1/(p + qx_j)$ and constructing the problem GLP_1, $max\ y$, with condition $x \in S$, shows that:

— the optimal solution to GLP_1 will be a feasible solution to GLP_0, if there is a feasible solution to GLP_0;

— any feasible solution to GLP_0 is a feasible solution to GLP_1 with $y \geqslant [r/(p + qL)] + (k - r)/p$ where L is such that $x_j \leqslant L$, $x_j \in S$.

Cabot's results can be obtained [6] also using the function $\sigma(x, \alpha) = \sum_{j=1}^{k} x_j/(\alpha_j + x_j)$ introduced by us. Indeed, let be the problem GLP_2:

$$\min \sigma(x, \alpha) = \sum_{x_j \in V_k} x_j/(\alpha_j + x_j), \quad x \in S.$$

We have:

— There exist a $\alpha > 0$ such that the optimal solution to problem GLP_2 is a solution having the least number of strictly positive elements in V_k, and hence is a feasible solution to problem GLP_0, if problem GLP_0 has a feasible solution.

— any feasible solution to GLP_0 is also a solution feasible to GLP_2 with $\sigma(x, \alpha) \leqslant r$.

As concerns the cardinality constrained linear programming problem (CCLP) [18] [20],

max cx

$$\text{subject to } Ax \leqslant b, \ x \geqslant 0, \ |x|^+ \leqslant r,$$

where $|x|^+$ denotes the cardinal of x i.e. the number strictly positive elements of x, it is shown in [6] that a solution of the problem: *max cx* with constraints $Ax \leqslant b$, $x \geqslant 0$ and $\sum_{x_j \in V_k} x_j/(\alpha_j + x_j) \leqslant r$ is a solution to (CCLP).

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SUR LA REPARTITION DE LA PRODUCTION PAR OUTILLAGE A L'AIDE DE LA PROGRAMMATION MATHEMATIQUE

Les auteurs ont introduit en [6] la fonction $(x, \sigma) = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} [p_j x_j/(\alpha_j + x_j)]$, $x \geqslant 0$, $\alpha \geqslant 0$, $p \geqslant 0$ en tant qu'alternative continue pour la fonction objectif à variables mixtes en vue d'optimiser l'activité d'analyse de la production et de la répartition, et tenant compte des coûts de démarrage de l'activité. Les auteurs ont construit quelques modèles qui ont pour base ce genre de fonction objectif concernant la répartition de la production par outillage, supposant que les restrictions des problèmes majeurs peuvent être mises en modèle linéaire.

К ВОПРОСУ О РАСПРЕДЕЛЕНИИ ПРОИЗВОДСТВЕННЫХ ЗАДАНИЙ ПО МАШИНАМ ПРИ ПОМОЩИ МАТЕМАТИЧЕСКОГО ПРОГРАММИРОВАНИЯ

Авторами поставлено в [6] функция $\tau(x, \alpha) = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} [p_j x_j / (\alpha_j + x_j)]$, $x \geqslant 0$, $\alpha > 0$, $p \geqslant 0$ как непрерывная альтернатива для целевой фукции со смешанными переменими оптимизации анализа производства и распределения с учётом затрат спуска в фабрикацию.В настоящей статье приведены некоторые модели, построенные авторами, имеющие на основе такой тип целевой функции, относящейся к распределению продукции по машинам, предпогагая, что ограничения основных проблемм могут быть модел ированы линейно.