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Applications of the Fractional Programming

The object of this paper is to point out some possibilities of using fractional programming in modelling economic processes. A series of examples have been compield to this end, from the vast speciality literature, devoted to fractional programming [39]. *Most applications are taken from the economic field. Nevertheless we shall point also to some applications from other fields as, for instance, from physics (optics) or those which occur even when solving some problems of mathematical programming.*

A series of practical optimization problems take into account the minimizing of a ratio of physical or economic functions, or both, as for instance cost/time, cost/volume, cost/benefit or other quantities which measure the efficiency of a system. Problems of this kind appear in various contexts, e.g. when minimizing the unit production cost of a certain product, i.e. the quotient between a (linear or non-linear) function, representing the total production cost, and a function representing the achieved quantity of that product. Such models, wherein function quotients appear in the objective function, make out *fractional programming problems* and they are of great interest for the non-linear programming as with the help of non-linear optimizing models a series of practical problems can be better described. In general, such problems are met with in problems with several quantitative criteria and for which the optimizing of a qualitative criterion is looked for, a criterion that appears like a ratio of quantitative criteria.

J.R. Isbell and W.H. Marlow [25] identified with priority fractional programming problems, deriving from military strategy problems. The fractional programming problems are particularly useful in solving economic problems in which various activities use certain resources in certain proportions and the objective is to optimize a certain indicator, as a rule the most favourable ratio of return and allocations, submitted to the restriction concerning the goods availability.

1. *Blending problems.* E. Munteanu and F. Radó consider in [31] the problem of determining, through fractional programming, the batches of the iron cast smelting furnaces, so as to obtain hot cast iron of the desired composition with minimal expenses, when scraps with no price are available. The paper [31] makes up one of the first fractional programming works, though the literature considers that the first work, on a world scale, belongs to A. Charnes and W.W. Cooper [11].

Let us consider that there are n raw materials, i.e.: M_1 ,..., M_p — ordinary pig iron sorts, M_{p+1}, ..., M_q — special pig iron sorts, M_{q+1}, ... M_{n-2} — past iron scrap, M_{n-1} — steel scrap, M_n — own scrap, in order to obtain a hot cast iron of certain composition. The data can be followed up in the table further down; two percentage limits have been proposed for each element (p_i, p_i'):

Raw material	1 kg. raw material contains				Value of a kg. of raw material
	Si (kg)	Mn (kg)	P (kg)	S (kg)	
M_j $(j = 1, ..., n)$	a_{1j}	a_{2j}	a_{3j}	a_{4j}	c_j
Hot cast iron %	1 kg. hot cast iron contains				
	p_1, p_1'	p_2, p_2'	p_3, p_3'	p_4, p_4'	

The Si, Mn, P, S contents of the batch are differing from those of the hot cast iron, due to a series of factors (kind of furnace, of fuel, smelting time, air delivery, melting temperature, a.s.o.). Thus, if b_i and b_i' $(i = 1, ..., 4)$ respectively, are the amounts of C, Si, Mn, P, S contained in a kg of batch, the following empiric formulas are valid:

$$p_i = \alpha_i + \beta_i b_i, \quad p_i' = \alpha_i + \beta_i b_i', \quad \alpha, \beta_i' = \text{const.}$$

If we consider as x_j the amount from the raw material M_j being present within a kilogramme of hot cast iron, we arrive to a fractional programming model:

$$\min_{x \in D} F = \frac{\sum_{j=1}^{n-1} c_j x_j}{\sum_{j=1}^{n-1} x_j} = \frac{\sum_{j=1}^{n-1} c_j x_j}{1 - x_n}$$

$$D = \left\{ x \Big/ \sum_{j=1}^{n} x_j = 1, \quad b_j \leqslant \sum_{j=1}^{n} a_{ij} x_i \leqslant b_i', \quad i = 1, 2, 3, 4; \quad x_j \geqslant 0, \; j = 1, ..., n \right\}$$

2. *Optimal materials cutting-on problem.* We shall consider the case of bar-shaped materials (steel bars, pipes, boards, paper rolls, etc.). We suppose, so as to simplify, they have identical lengths and cross-sectional areas. The material must be cut, according to a certain order, in pieces of various lengths, so as to obtain at least n_i pieces of length l_i $(i = 1, ..., m)$. It is ob-

vious that in order to have a solution to the problem, we must have $l_i \leqslant l$, where l is the length of the bars.

The problem consists in achieving the order with a minimal number of cut bars. Considering a bar of length l, there are several possible modes of cutting bars of lengths l_i. We call cutting-on pattern (or cutting-on solution) a particular way of cutting a bar, e.g. a_1 bars of length l_1, a_2 bars of length l_2, etc. Let us suppose that all the n distinct cutting solutions have been found out, through a certain procedure. We denote by x_j the number of usages of pattern j, by a_{ij} the number of bars of length l_i, which are cut according to pattern j. The cutting-on problem is stated as follows:

$$min\left\{F = \sum_{j=1}^{n} x_j \,\middle|\, \sum_{j=1}^{n} a_{ij}x_j \geqslant n_i, \quad x_j \text{ integer and positive}\right\}$$

The problem is particularly complicated, due to the fact that, in actual situations n is very big, especially due to the condition that x_j must be an integer. In [24] it is evinced that, within practical conditions, a variation field $[n_i', n_i'']$ is admitted for n_j, and no requirement that n_j be integer is compulsory. In this case, the wastes percentage is taken as an objective. The wastes derived from using pattern j is

$$w_j = l - \sum_{i=1}^{n} l_i a_{ij}$$

When using x_j times the pattern j, we obtain a total percentage of wastes equal to

$$100\left(\sum_{j=1}^{n} w_i x_i / l \sum_{j=1}^{n} x_j\right)$$

In this way, the cutting-on problem is formulated as a fractional programming problem:

$$min\, F = \frac{\sum_{j=1}^{n} w_j x_j}{\sum_{j=1}^{n} x_j}$$

$$n_i' \leqslant \sum_{j=1}^{n} a_{ij}x_j \leqslant n_i'' \quad i = 1, \ldots, m$$

$$x_j \geqslant 0 \quad j = 1, \ldots, n.$$

3. *Information theory.* B. Meister and W. Oetlli [30] and S.P. Aggarwal and I.C. Sharma [1] applied the fractional programming for computing the maximal transmission rate through an information channel.

Let us take a constant and discrete transmission channel, with m input and n output symbols, characterized by a transition matrix P:

$$P = ((p_{ij})), \quad i = 1, \ldots, n; \quad j = 1, \ldots, m$$

$$p_{ij} \geqslant 0, \quad \sum_i p_{ij} = 1$$

Here p_{ij} represents the probability to obtain at the output symbol i, with the condition that symbol j was emitted at the input. We suppose, as a rule, that matrix P has not lines equal to zero, and this means that no output symbols exist which will never be received.

A cost (energy, time duration) $t_j > 0$ is associated to the input symbol transmission. Denoting by $x = (x_j)$ the input probability distribution, where $x_j \geqslant 0$ and $\sum_j x_j = 1$, the transmission rate is defined thus:

$$T(x) = \frac{\sum_i \sum_j x_j p_{ij} \log\left(p_{ij} \Big/ \sum_k x_k p_{ik}\right)}{\sum_j t_j x_j}$$

The relative capacity C of the channel is defined as being the maximum of $T(x)$ over all admissible input distributions x. We are led, this way, to a fractional programming problem.

4. *Minimum risk problems in the stochastic programming.* The minimum risk problems in the stochastic programming have been considered for the first time by B. Bereanu [7], [8] and A. Charnes and W.W. Cooper [13].

Let us consider the linear programming problem

$$\max_x \{c'x \mid Ax = b, \quad x \geqslant 0\} \tag{1}$$

which frequently appeares in the current planning of enterprises, when intending to maximize the various factors of their economic activity. Such indicators, as for instance the total hoarding money, may be considered as being the maximum of a linear programming problem. In the case when random elements appear in the linear programming problem, it is natural to select, from the admissible plans, that having a maximal probability that the achieved indicator be at least equal to the planned one.

Coming back to problem (1), if c is a random vector , a level k of resources and expenditures is taken, and thereafter the probability that the objective function takes at least value k *is* maximized:

$$\max_{x \in D} P\{\omega \mid c'(\omega)x \geqslant k\}$$

In the hypothesis that costs c_j are independent random variables, having $\bar{c}$ as mean values vector and V as covariance matrix, the determination of the minimum risk solution is equal to solving the fractional programming problem:

$$\max\left\{\frac{\bar{c}'x - k}{\sqrt{x'Vx}} \,\middle|\, Ax = b; \quad x \geqslant 0\right\}$$

5. *Macroeconomic planning.* One of the most significant applications of the fractional programming is to be found in linear macrodynamic models of national planning based on the input-output analysis.

Let $Y(t)$ be the national income, created in year t and $Y_i(t)$ the national income created in branch i in year t. It is obvious that $Y(t) = \sum_i Y_i(t)$.

If we denote by $C_{ik}(t)$ the consumption of type i products (created in branch i) in branch k, and by $I_{ik}(t)$ the share of the national income created in branch i, meant for investments in branch k, then the following equation of distribution of the national income, created in branch i, is valid:

$$Y_i(t) = \sum_k (C_{ik}(t) + I_{ik}(t)).$$

The growth of the national income in branch i depends on the investments done in that branch:

$$\Delta Y_k(t) = Y_k(t+1) - Y_k(t) = b_k I_k(t) + b_{ko}$$

where:

$$I_k = \sum_i I_{ik}$$

In these conditions, the macroeconomic planning consists in maximizing the growth rate of the national income

$$max \frac{\Delta Y}{Y} = \frac{\sum_k (b_k I_k(t) + b_{ko})}{\sum_i \sum_k (C_{ik}(t) + I_{ik}(t))}$$

with restrictions

$$C_k(t) \geqslant max\ (\bar{C}_k,\ C_k(0)); \text{ where } C_k(t) = \sum_i C_{ik}(t)$$

$$I_k(t) \geqslant I_k(0)$$

$$I_k(t) \leqslant I_k(max)$$

where $\bar{C}_k$ represents the minimal consumption pre-assigned to sector k and $I_k\ (max)$ the maximal investment level allotted by the society for sector k.

The above problem makes out a kind of macroeconomic planing problem.

6. *Production efficiency problems.* S. P. Bradley and S.C. Frey, Jr. consider in [10] the example of maximizing the ratio profit/cost, wherein the profits represent a Cobb-Douglas-type production function. We mean through production function (which is a concept used in econometry) the relation between the variables which designate the production factors (inputs) and the production results (outputs). Let us consider the value of the profit corresponding to a set of projects ($i = 1, ..., m$), given through a Cobb-Douglas -type production function:

$$B = \prod_{j=1}^{n} a_0 x_j^{a_j} \text{ where } \sum_{j=1}^{n} a_j = 1$$

We denoted by x_j the production factors and let the total cost be a linear function of the levels of investment in various projects:

$$C = \sum_{j=1}^{n} c_j x_j + c_0$$

The maximizing of the ratio profit/cost with restrictions concerning the expended funds is required. The ratio profit/cost is an efficiency indicator, i.e. a ratio between what is obtained and what is allotted. We get the following production efficiency model:

$$max\left\{ \frac{\prod_{j=1}^{n} a_0 x_j^{a_j}}{\sum_{j=1}^{n} c_j x_j + c_0} \;\middle|\; \sum_{j=1}^{n} a_{ij} x_j \geqslant b_i,\ i = 1, ..., m,\ x_j \geqslant 0,\ j = 1, ..., n \right\}$$

where b_i represents the profit which can be achieved by project i and a_{ij} represents the contribution of the production factor j within project i, in order to achieve profit b_i.

7. *Mathematical taxonomy.* Some problems of classification of a set of elements in a given number of classes lead to fractional programming models. Let A be a set formed of N elements and let M be the wanted number of classes wherein the elements of the set have to be divided. We admit that for any pair (i, j) the distance d_{ij} between entities i and j was defined. We require to divide the entities of set A in M classes, so that the totals of the means of the distance squares sum within the groups be minimal.

We define the Boolean variables as follows:

$$x_{ij} = \begin{cases} 1 \text{ if entity } i \text{ belongs to class } C_j \\ 0 \text{ for the remainder.} \end{cases}$$

The problem of the determination of an optimal division into classes of the entities of set A is equivalent to the following *non-linear Boolean fractional programming problem*,

$$min \sum_{k=1}^{M} \frac{\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d_{ij}^2 x_{ik} x_{jk}}{\sum_{i=1}^{N} x_{ij}}$$

$$\sum_{k=1}^{M} x_{ik} = 1, \qquad i = 1, 2, ..., N$$

$$x_{ik} = 0 \text{ or } 1;\ i = 1, ..., N;\ k = 1, ..., M.$$

The reader finds in [35] also other classifying problem types, which can be modelled through fractional programming.

8. *Information processing. The fractional programming finds applications* not only in the economic field , but also in other fields, as for instance

in physics [20]. Within the spectral filters for the quadratic law detector for infrared radiations, there appears the problem of maximizing the ratio signal/noise. From the mathematical point of view, the problem consists in maximizing the filter-function

$$\Phi(x) = \frac{(a'x)^2}{x' Bx + \beta}$$

in the field

$$S = \{x \in E^n, \ 0 \leqslant x_i \leqslant 1, \ i = 1, ..., n\}$$

where α and β are strictly positive constants, B is a symmetrical positively defined matrix, $(a'x)^2$ represents the incoming signal, $x' Bx + \beta$ the variance of the background signal. The constraint S illustrates the fact that the filter can not transmit more than 100% and no less than 0% of the total energy. The problems of optical filtering are particularly topical, through their prospects of optically processing the information. This made it possible to pass over to the utilization of optical phenomena in the optical computers for information processing.

9. *Optimal path problems in graphs.* V. Peteanu and Şt. Ţigan [33] come to a fractional programming problem in relation to an optimal path problem, in twice weighed graphs.

Let $G = (X, U)$ be a convex graph, having no cycles, wherein $X = \{x_1, ..., x_n\}$ represents the set of arcs. Let a_{ij} and b_{ij} two positive weights associated to each arc $(x_i, x_j) \in U$ and be $\mathfrak{L}(G)$ be the set of pathes from knot x_1 to knot x_n. A path $L^* \in \mathfrak{L}(G)$ must be determined, such that

$$min \left\{ \frac{\sum\limits_{(x_i, x_j) \in L} a_{ij} + \alpha}{\sum\limits_{(x_i, x_j) \in L} b_{ij} + \beta} \middle| L \in \mathfrak{L}(G) \right\} = \frac{\sum\limits_{(x_i, x_j) \in L^*} a_{ij} + \alpha}{\sum\limits_{(x_i, x_j) \in L^*} b_{ij} + \beta}$$

where α and β are given real numbers, which constitutes a fractional programming problem on a finite set (set of pathes).

This problem was previously dealt with by G.B. Dantzig, W. Blattner and M.R. Rao [16] and B. Fox [22] with the difference that L is a circuit and a_{ij} and b_{ij} respectively have the signification of a cost or of a time associated to arc (x_i, x_j), respectively.

The work [33] mentions also another type of fractional problem, namely the determination of a path $L^* \in L(G)$ so that:

$$min \left\{ \frac{\max\limits_{(x_i, x_j) \in L} \{a_{ij}\}}{\max\limits_{(x_i, x_j) \in L} \{b_{ij}\}} \middle| L \in \mathfrak{L}(G) \right\} = \frac{\max\limits_{(x_i, x_j) \in L^*} \{a_{ij}\}}{\max\limits_{(x_i, x_j) \in L^*} \{b_{ij}\}}$$

10. *Distribution problems.* In the case of the resources dynamic distribution problem the maximizing of the total income achieved through integrally using the resources is taken into account. We can also consider another mode of optimally distributing the resources i.e. the one wherein

the resources utilization efficiency (mean income by time unit) is maximized. Acting this way, Ştefan Ţigan [43] obtains the following *separable fractional programming model*

$$max\ F(x) = \frac{\sum_{i=1}^{n} f_i(x_i) + \alpha}{\sum_{i=1}^{n} g_i(x_i) + \beta}$$

$$\sum_{i=1}^{n} w_i x_i \leqslant b$$

$$\alpha_i \geqslant x_i \geqslant \beta_i \geqslant 0, \quad x_i \text{ integers}$$

where w_i, α_i, β_i $(i = 1, ..., n))$, α, β and b are given real numbers.

11. *Transport problems with several objective functions.* The fractional programming can become a way of solving the problem of mathematical programming with several objective functions. For many such problems the criteria concordance is not obvious, certain criteria being even contradictory. Among those we mention the transport problem having three efficiency functions: the afferent cost, the total transport time which must be minimal and the quantity to be transported which must be maximal. If we denote by x_{ij} the quantity to be transported, the mathematical model of the problem of transport with limited capacity is the following one:

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, \ldots m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, ..., n \tag{3}$$

$$0 \leqslant x_{ij} \leqslant g_{ij} \quad i = 1, ..., m;\ j = 1, ..., n \tag{4}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{5}$$

$$min\ F_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{6}$$

$$min\ F_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij}\, x_{ij} \tag{7}$$

$$max\ F_3 = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \tag{8}$$

In paper [40] the author considers that the respective problem can be solved by constructing a synthesis function of the form:

$$F = \frac{F_1 F_2}{F_3^2} = \frac{\left(\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij} x_{ij}\right)}{\left(\sum_{i=1}^{m}\sum_{j=2}^{n} x_{ij}\right)^2} \tag{9}$$

which must be minimized.

It is proved that through solving the mathematical programming problem with restrictions (2) — (5) and the objective function (9), efficient solutions for the stated problem are obtained.

12. *Investment allocation problems.* R.T. Ziemba, F.J. Brooks-Hill and C. Parkan [44] arrive to a fractional programming model by considering the optimal selection problems in the case of investing a certain sum in n productive activities. Let us denote by x_i the percentage of the respective fund, which must be invested in activity i. The vector $x = (x_1, ..., x_n)$ will satisfy restrictions $\sum_{i=1}^{n} x_i = 1$, $x_i \geqslant 0$ to which other restrictions obtained through economic considerations may be added. If ξ_i is the income obtained when investing the whole sum in activity i, then the income corresponding to the investments policy x is $V(x) = \sum_{i=1}^{n} \xi_i x_i$.

The problem becomes highly complicated, as ξ_i are not constants but rather random variables. In the hypothesis that ξ_i are normals, with means $\overline{\xi}_i$ and covariance matrix $\Sigma = (\sigma_{ij})$ (positively defined matrix), then the mean value of the total income is $E\ (V(x)) = \sum_{i=1}^{n} \overline{\xi}_i x_i$ and the mean square value deviation $D(V(x)) = \sqrt{x' \Sigma x}$.

By considering both functions together (i.e. the minimizing of the total income mean value and the minimizing of its mean square value deviation), one arrives to the following fractional programming problem [44] :

$$max\ F(x) = \frac{\sum_{i=1}^{n} \overline{\xi}_i x_i}{(x' \Sigma x)^{1/2}} = \frac{\overline{\xi} x}{(x' \Sigma x)^{1/2}}$$

with restriction $\sum_{i=1}^{n} x_i = 1$, $x_i \geqslant 0$, $x \in S$ where S results from other restrictions of economic character.

13. *Investment territorial distribution problems.* In connexion with the previous model, we present here another example considered by E. Trattner and C. Zidăroiu [42], as concerns the socio-economic efficiency of investments, in terms of the cost of creating new jobs. Let us suppose that in a certain county the problem is set to achieve m investments whose objective is to attract into industry an important number of farm workers. Let s be

the number of modules according to which the investment $i(1 \leqslant i \leqslant m)$, can be carried out, I_{ij} the i-type investment cost, in the case when it is achieved according to module $j(1 \leqslant j \leqslant s_i)$, f'_{ij} (or f''_{ij} respectively) labour necessary in the time unit, F the farming labour in the county, which can be attracted into industry. The module concept can be understood when speaking of some versions of locating the objectives within the county.

If we denote by x_i the necessary labour in the industrial units $i(i = 1, ..., m)$ and by y_{ij} a Boolean variable defined as follows:

$$y_{ij} = \begin{cases} 1, & \text{if investment } i \text{ is achieved with module } j \\ 0, & \text{in the opposite case} \end{cases}$$

the mathematical model is the following one:

$$\min \frac{\sum_{i=1}^{m} \sum_{j=1}^{s_i} I_{ij}}{\sum_{i=1}^{m} x_i} \tag{10}$$

$$\alpha_1 F \leqslant \sum_{i=1}^{m} x_i \leqslant \alpha_2 F \tag{11}$$

$$\sum_{j=1}^{s_i} f'_{ij} y_{ij} \leqslant x_i \leqslant \sum_{j=1}^{s_i} f''_{ij} y_{ij}, \quad 1 \leqslant i \leqslant m \tag{12}$$

$$\sum_{j=1}^{s_i} y_{ij} \leqslant 1, \quad 1 \leqslant i \leqslant m \tag{13}$$

$$x_i \geqslant 0; \quad y_{ij} = 0 \text{ or } 1 \qquad \begin{matrix} 1 \leqslant i \leqslant m; \\ 1 \leqslant j \leqslant s_i \end{matrix} \tag{14}$$

The efficiency function (10) represents the mean cost for creating through investments a new job in a given county. Consequently, the problem of the optimal investment selection, in terms of the cost of creating new jobs, reduced itself to a mixed model with linear restrictions and a linearly fractional objective function.

14. *Problems of decision on Markov chains.* [17], [21]. Certain problems of optimal control of the dynamic systems can be modelled through fractional programming. Let us consider a controlled dynamic stochastic system, which can be find itself, at certain moments $t = 0, 1, ...$ in one of the states $0,1, ..., L$. The system is directed from a state to another state, through one of the decisions $d_1, ..., d_k$, which will determine the probabilities law of changing the system. We denote by $X_0, X_1, ...$ the states of the system at moments $t = 0. 1, ...,$ and by $\Delta_0, \Delta_1, ...$ the corresponding decisions and by C the class of all decision rules R consisting of functions:

$D_k(X_0, \Delta_0, ..., X_t) = P(\Delta_t = d_k \mid X_0, \Delta_0, ..., X_t), \quad k = 1, ..., K; \quad t = 0,1, ...$

It is obvious that $D_k(X_0, \Delta_0, ..., X_t) \geq 0$, $k = 1, ..., K$

$$\sum_{k=1}^{K} D_k(X_0, \Delta_0, ..., X_t) = 1$$

We suppose that

$$P(X_{t+1} = j \mid X_0, \Delta_0, ..., X_t = i) = \sum_{k=1}^{K} q_{ij}(k)\, D_k(X_0, \Delta_0, ..., X_t = i)$$

$$i, j = 0 ..., L; \quad t = 0, 1, ...$$

where $\{q_{ij}(k)\}$ are stochastic matrices for each k.
We consider $C' \subseteq C$ the decision rules class so that:

$$D_k(X_0, \Delta_0, ..., X_t = i) = D_{ik}, \quad i = 0 ,..., L; \quad k = 1, ..., K.$$

i.e. the class of those decision rules which no longer depend on X_0, Δ_0, ..., X_{t-1}, Δ_{t-1} and t but only on the last state and the adopted decision . We can show [17] that it is sufficient to limit ourselves to the C' class decision rules neglecting those from classes $C-C'$. For $R \in C'$ the sequence $\{X_t\}$, $t = 0, 1, ...$ is a Markov chain with stationary transition probabilities $\{p_{ij}\}$ where:

$$p_{ij} = \sum_{k=1}^{K} q_{ij}(k)\; D_{ik}, \; i, j, = 0, ..., L.$$

A cost is associated to each state and to each decision rule. Let $W_{ik}(t) > > 0$ be the estimated cost assigned at moment t, being given the fact that the system finds itself in state i and decision d_k is applied to it, and let W_t be the expected cost assigned at moment t, as a consequence of the application of decision rule R. We suppose that $W_{ik}(t) = W_{ik}$.
The following problem may be formulated: If $P\ (X_0 = i) = 1$ and for each $R \in C'$, the states $0, ..., L$ belong to the same class, as concerns the classification of the states of a Markov chain, let us determine that decision rule $R^* \in C$ so that

$$\psi_{R^*}(i) = \min_{R \in C} \psi_R(i), \quad i = 0, ..., L \tag{15}$$

where

$$\psi_R(i) = \lim_{T \to \infty} Sup \frac{\sum_{t=0}^{T} W'_t}{\sum_{t=0}^{T} W''_t} \tag{16}$$

In other words, we take as minimizing criterion the ratios of the mean costs by time unit, concerning two sets of costs W'_{ik}, W''_{ik}. We can prove

that, when $R \in C'$ then

$$\psi_R(i) = \frac{\sum_{i=0}^{L} \sum_{k=1}^{K} \pi_i D_{ik} W'_{ik}}{\sum_{i=1}^{L} \sum_{k=1}^{K} \pi_i D_{ik} W''_{ik}} \quad i=0, ..., L \tag{17}$$

where π_i $(i = 0, ..., L)$ satisfies the relations

$$\pi_j - \sum_{i \in I} \pi_i p_{ij} = 0, \quad j \in I \text{ (the Set of States)}$$

$$\sum_{j \in I} \pi_j = 1, \quad \pi_j \geqslant 0$$

Consequently, the determination of a decision-making rule which minimizes a cost criterion, means the same as solving a fractional programming problem.

15. *Equipment replacing problem* [27] The issues presented in the previous example can be applied in determining the optimal inspection, maintenance and replacing policies, for the systems whose impairment may be described as a finite Markov chain with a discrete time parameter. Let us consider an equipment able or not to run, which can have an impairment, stated after an inspection. After inspection the equipment is either replaced or kept further. In the latter case, either the necessary repairs are done, or the subsequent inspection is established. We suppose that the impairing law of the system (equipment) is a Markov-type one, i.e. the series of succesive states from a discrete Markov chain, with stationary transition probabilities q_{ij} $(i, j = 0, ..., L)$. This supposition is plausible for the concrete cases, for instance with the electronic systems (installations) with several components, every component having (approximalety) an exponential wear, and one can expect that the whole system has a Markov description. The mathematical model is as follows:

$$min\, F = \frac{\sum_{i} \sum_{s,k} x_{isk} h_{isk}}{\sum_{i} \sum_{s,k} k x_{isk}} \tag{18}$$

$$\sum_{s,k} x_{jsk} = \sum_{i} \sum_{s,k} x_{isk} V_{skj}, \quad j = 0, 1, ..., L(K-1) \tag{19}$$

$$\sum_{j} \sum_{s,k} x_{isk} = 1 \quad k = 1, ..., K \tag{20}$$

16. *Games theory problems.* Fractional functions appear when solving certain games. R.G. Schroeder [37] considers the determination of minimax solutions to a game whose payoff matrix is of the form:

$$F(X, Y) = X'AY/X'BY \tag{21}$$

where A and B are real matrices $(B > 0)$ and X and Y are mixed strategies

Such situations appear, for instance, if we consider a stochastic game where the matrix A contains payoffs for each move and B is a matrix of the probabilities that the game comes to an end before the next move. With each move the players must take into account, besides the immediate payoff, also the probability of payoffs on future moves. Then $F(X, Y)$ will represent the expected payoff for the entire sequence of moves:

$$F(X, Y) = \sum_{n=0}^{\infty} X'AY(1 - X'BY)^n = \frac{X'AY}{X'BY}$$

Functions of the form (21) appear within games with nonzero-sum. If A is the payoff matrix for the first player and B for the second player, then one player tries to maximize the ratio of payoffs, while his adversary will minimize this ratio.

17. *Stocks problem.* We shall consider a simple case, that is to say the one of a single product and of a single level. In a time period T, a demand R with a constant intensity $r = R/T$ must be satisfied. The production rate (k) is supposed to be constant ($k > r$). Initially, (at zero moment) the stock is zero and it increases continuously during a time period t_1 up to the maximal value S, then it decreases continuously down to zero during a time period t_2. It follows a time period t_3 wherein non-satisfied demands accumulate up to the maximal deficit s. At the end of period t_3 the production begins again and the remainders diminish untill they are completely eliminated after a period t_4. The cycle resumes itself. The mean expenditures by time unit are:

$$F(t_1, t_2, t_3, t_4) = \frac{\frac{1}{2}[c_1 S(t_1 + t_2) + c_2 s(t_3 + t_4)] + c_3}{t_1 + t_2 + t_3 + t_4}$$

where c_1 represents the stocking cost of a unit of the considered product in the time unit; c_2 — the scarcity cost in the time unit; c_3 — the fixed expenses (for a production cycle). The above mentioned function can be brought to the form:

$$F(t_2, t_3) = \frac{\frac{1}{2} kr(c_1 t_2^2 + c_2 t_3^2) + c_3(k - r)}{k(t_2 + t_3)}$$

The problem is how to determine t_2 and t_3, so that the fractional function $F(t_2, t_3)$ be minimal.

18. *The analysis of the sensitivity of the solution of a linear programming problem.* This problem, which means a particularly important field in the mathematical programming, consists in studying the variation of the optimal solution of a mathematical programming problem, in general for varying the problem's data.

In what follows we shall refer to the repricing theorem, considered by A. Charnes and W.W. Cooper [12] for testing the sensitivity of a linear

programming model, a problem which leads to a fractional programmin model. We consider an ordinary transport problem.

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} \tag{22.a}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, ..., m \tag{22.b}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, ..., n \tag{22.c}$$

$$x_{ij} \geqslant 0, \quad i = 1, ..., m \quad j = 1, ..., n \tag{22.d}$$

Let $\{x_{ij}^*\}$ be its optimal solution and Z^* the minimal transport cost. There is a possibility that, by modifying prices c_{ij}, Z^* be also modified.

The work [12] shows that Z^* is a concave function of c_{ij} and it is finitely piecewise linear in c_{ij}. For simplifying, we pay attention to a single price, for instance c_{pq}. We intend to determine how big this price can be (the interval wherein it can be found), so that the transport cost Z^* decreases by at least kZ^* (k is a proportionality factor $0 < k < 1$). The problem becomes: $max\ c_{pq}$ with constraints (22.b) — (22.d) and the constraint:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + c_{pq}x_{pq} \leqslant Z^*(1-k),$$

where the indices p and q are excluded within the double summing up.

In this model the unknowns will be x_{ij} and c_{pq}. This problem is nonlinear because we have in the constraints the product of two variables. Effecting the change of variables $y_{pq} = c_{pq}x_{pq}$ the nonlinearity can be transferred to the objective function. The above model becomes a fractional programming model: $max\ \frac{y_{pq}}{x_{pq}}$ with the constraint $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + y_{pq} \leqslant Z^*(1-k)$ and the constraints (22.b) — (22.d).

19. *Marine transport problems* [9]. Let us suppose that in a port A we have to load a ship (of limited capacity) with a load formed up of goods of various types, as concerns their weights and values, goods which must be transported to another port B. It is required to determine how much of each merchandise has to be loaded so that the load of the ship leads to a maximal value profit.

Let:

— Q_i be the maximum tonnage available of the cargo of type $i (i = 1, ..., N)$;

— V be the maximum weight supported by the ship;

— f_i be the benefit obtained through loading a unit of the i-th type of merchandise (monetary units/ton);

— g_i be the weight of the i-th type of merchandise.

Denoting by x_i the number of articles from the i-type merchandise, which are loaded, the mathematical model of the loading problem is the following one:

$$max\left\{\sum_{i=1}^{N} f_i x_i \mid x_i \leqslant Q_i, \quad \sum_{i=1}^{N} g_i x_i \leqslant V, \quad x_i \geqslant 0\right\}.$$

When considered in this way, the problem is not correctly modelled through the above model in most cases. As the loading time differes from a ship to another one, we must take also into account the stay of the ship in the port, as this will influence the transport cost. Let y_i be the daily loading rate (tons/day) of merchandise i, c_1 the daily cost of the stay of the ship in the port, c_2 the daily cost of the ship on sea, s its displacement speed, and d the total distance. Thus, the total time for a transport will be:

$$T = \frac{d}{s} + \sum_{i=1}^{N} (x_i/y_i) \text{ and the benefit of a transport will be:}$$

$$P = \sum_{i=1}^{N} f_i x_i - \sum_{i=1}^{N} [c_1 x_i/y_i - c_2(d/s)].$$

It would be more realistic to consider the mean profit by time unit (day, month, year), hence to optimize a function of the form:

$$F = \frac{P}{T} = \frac{\sum_{i=1}^{N} f_i x_i - \sum_{i=1}^{N} (c_1 x_i/y_i) - c_2(d/s)}{\frac{d}{s} + \sum_{i=1}^{N} x_i/y_i}$$

20. *Optimal transport plan* [3]. An optimal loading-unloading plan has to be set out, for a ship which sails through a route linking N ordered ports. The ship makes $M (M \leqslant N)$ stopovers (representing disjoint sets of ports) and which keep the order between the sets) for loading and unloading. In every port, the loads are available to be transported to the next ports. The load amounts available at each stopover are random variables, independent and defined on a finite set of discrete values. To each stopover m ($m = 1, \ldots, M$) one knows which were the decisions at the precedings stopovers and the available loads. The problem consists in determining the amounts and the places of destination of the goods which will be loaded in the ports of this stopover, so that the mean of the incomes sum by unit of time with the next stopovers of the route be maximized. The objective function appears in the present case as the sum of some functions, linearly fractional and separable. We make the hypothesis that the loading/unloading times and costs depend linearly on the quantities and that a single type of load is transported from a port to another one.

Let Q_{ij} (or S_{ij} respectively) be the amount of available load shipped from port i to port j in percentages from the ship capacity

$$0 \leqslant Q_{ij},\ S_{ij} \leqslant 1, \quad i = 1, \ldots, N;\ j = i + 1, \ldots, N + 1$$

Let T_{ij} be the time required to load completely a ship in port i and to unload it in port j, calculated in percents from the total time (time on sea and time in ports), consumed to cover the subroute which represents the set to which port i belongs to.

Let R_{ij} be the net income resulting from the transportation of a unit from port i to port j, devided by the time required to load and unload this unit.

Let C_m be the difference between the costs by unit of time on sea and in the port multipled by the ratio between the time on sea and the total time on subroutes, representing the set m; $C_m > 0$, $m = 1, ..., M$.

$P_m = \{i \mid i \text{ is in the set } m \text{ of indices}\}$, $m = 1, ..., M$

$\bar{Q}^m = \{\bar{Q}_i \mid i \in P_m\}$, $\bar{S}^m = \{\bar{S}_i \mid i \in P_m\}$.

The problem can be considered as a decision process in M stages. The mathematical model for stage m is as follows:

$$max\ f_m(\bar{Q}^m) = \frac{\sum_{i \in P_m} \sum_{j>i} T_{ij} R_{ij} Q_{ij} - C_m}{\sum_{i \in P_m} \sum_{j>i} T_{ij} Q_{ij} + 1}$$

$$\sum_{j>i} Q_{ij} \sum_{p<i} \sum_{q>i} Q_{pq} \leqslant 1 \ \forall\ i$$

$$0 \leqslant Q_{ij} \leqslant S_{ij} \ \forall\ i, j$$

Similar problems can be found in [2] and [4].

21. *Indefinite programming problems.* The fractional programming problems appear when solving certain mathematical programming problems, for instance the indefinite programming.

K. Swarup [41] considers the indefinite quadratic programming problem

$$max\{(c'x + \alpha)\ (d'x + \beta) \mid Ax \leqslant b;\ x \geqslant 0\}$$

that he solves through another equivalent convex problem. Effecting the variable transformation $Y = tx$ with $t \geqslant 0$ selected so that $d'y + \beta t = \delta$ ($\delta = \text{const} \neq 0$) the previous problem becomes:

$$max\left\{\frac{c'y + \alpha t}{t^2} = \frac{1}{\dfrac{t^2}{c'y + \alpha t}} \;\middle|\; Ay - bt \leqslant 0,\ d'y + \beta t = \delta,\ y,\ t \geqslant 0\right\}$$

As function $\dfrac{t^2}{c'y + \alpha t}$ is a convex function, it results that we obtained a convex programming problem.

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APPLICATIONS DE LA PROGRAMMATION FRACTIONNELLE

Le but de cet ouvrage est celui de mettre en évidence quelques possibilités d'utiliser la programmation fractionnaire à la simulation des processus économiques. A cet effet, de nombreux exemples ont été compilés de la vaste littérature consacrée à la programmation fractionnaire [39]. Le plus grand nombre d'applications appartient au domaine économique. Nous signalerons pourtant aussi quelques applications appartenant à d'autres domaines tels que la physique (optique) ou celles qui interviennent dans la résolution même de certain problèmes de programmation mathématique.

ПРИЛОЖЕНИЯ ДРОБНОГО ПРОГРАММИРОВАНИЯ

Целью настоящей работы является выявление некоторых возможностеи использования дробного программирования при моделировании экономических процессов. Таким образом, был получен ряд примеров из обширной литературы, посвященной дробному программированию [39] Большинство приложений находится в экономике. Отмечаем также и некоторые приложения в физике (оптике) и в случаях разрешения проблем математического программирования.

29. I. 1979