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An Algorithm for the Solution of the Dynamic Production-Transportation Problem

The author presents the general dynamic production-transportation problem; on the basis of some substitutions, this problem is reduced to a linear programming problem of a special type. In the case in which there are no constraints concerning the storage capacities by the producers or by the consumers and, moreover, there exist no transportation routes with limited capacities, it is shown that the dynamic problem can be reduced to an ordinary problem of the transportation type. The algorithm of compared differences for the solution of transportation problems is presented, as well as the way in which this algorithm is applied to the solution of the dynamic production-transportation problem.

1. INTRODUCTION

It is well known that the model of the transportation problem stands for one of the linear programming models with broad applications in the optimization of numerous technical and economic phenomena.

In the paper we consider the case of m production points which realize a homogenous product. These production points have to satisfy the necessities of n consumers in the periods $t = 0, 1, ..., T - 1$, where T is fixed.

The maximum production capacity of every production point in every period and the necessities of every consumer in every period t ($t = 0, 1, ..., T - 1$) are considered given.

Other "known" elements of the model are the following: the unitary price of the product realized by every production point i ($i = 1, 2, ..., m$)

in every period t ($t = 0, 1, ..., T - 1$), the unitary transportation cost of the product from every production point to every consumer in every period t, the maximum storage capacity at every production point and every consumer, the unitary storage cost at every producer or consumer in every period t, as well as the maximum transprtation capacities between every production point and every consumer, in every period.

The problem consists in the determination of the production, storage and transportation plan which assures the minimum production, transportation and storage expenses for the whole planning period $[0, T]$.

In section 2, the formulation of the general dynamic model of the production-transportation problem is given.

By the introduction of new variables, this is transformed in a linear programming model of a special form (relations (10) — (16)). In the case in which there are no constraints imposed upon the storage and transportation capacities, one shows that the model becomes the model of an ordinary transportation problem (relations (25) — (28) or (25') — (28')).

Since problem (25) — (28) has many blocked routes, its matrix being, in fact, triangular, for its solution one suggests the application of the algorithm of compared differences, worked out by the author in 1961.

In section 3 the method of compared differences is presented and, in section 4, the algorithm for the solution of transportation problems deduced on the basis of this method.

In section 5 we present the method of solutioning the dynamic production-transportation problem without constraints on the storage or transportation capacities on the basis of the algorithm of compared differences.

2. THE MATHEMATICAL MODEL OF THE DYNAMIC PRODUCTION-TRANSPORTATION PROBLEM

One considers m production points, every point i having in period t the maximum production capacity $a_i(t)$ ($i = 1, 2, ..., m$; $t = 0, 1, ..., T - 1$).

These m production points have to satisfy n consumers, $b_j(t)$ standing for the $j - th$ consumer's demand in period t($j = 1, 2, ..., n$; $t = 0, 1, ..., T - 1$). The maximum storage capacity for period t at production point i is $\alpha_i(t)$, and consumer j it is $\beta_j(t)$.

The maximum transportation capacity between the production point i and the consumer j in period t is $\delta_{ij}(t)$.

The price of the product in period t at production point i is $p_i(t)$, the transportation cost for a unit of the product from production point i to the consumer j, in period t, is $c_{ij}(t)$, and the unitary storage cost in period t is $h_i(t)$ at production point i and $q_j(t)$ at consumer j, respectively.

The problem is to determine the optimum plan of production, storage and transportation, that is the plan which assures the satisfaction of the consumer's demand at every period t ($t = 0, 1, ..., T - 1$), minimizing the total expenses of production, storage and transportation during the whole period [0, T].

We introduce the following *decision parameters:*

$X_i(t)$ — the production of production point i in period t
$(i = 1, 2, ..., m;\ t = 0, 1, 2, ..., T-1)$;

$u_{ij}(t)$ — the quantity which is to be transported from production point i to consumer j in period t
$(i = 1, 2, ..., m;\ j = 1, 2, ..., n;\ t = 0, 1, ..., T-1)$.

We have the following *state variables*:

$z_i(t)$ — the volume of the stock existent at the beginning of period t at production point i
$(i = 1, 2, ..., m;\ t = 0, 1, ..., T)$.

$y_j(t)$ — the volume of the stock existent at the beginning of period t at cosumer j
$(j = 1, 2, ..., n;\ t = 0, 1, ..., T)$.

Without diminishing the generality of the problem, we may suppose that the initial stock is null, that is $z_i(0) = 0$ $(i = 1, 2, ..., m)$; $y_j(0) = 0$ $(j = 1, 2, ..., n)$, and the final stocks have to be nonnegative; $z_i(T) \geqslant 0$ $(i = 1, 2, ..., m)$; $y_j(T) \geqslant 0$ $(j = 1, 2, ..., n)$. Moreover, T is considered to be fixed.

With the above notations, the mathematical model of the dynamic production-transportation problem is the following;

$$z_i(t+1) = z_i(t) + X_i(t) - \sum_{j=1}^{n} u_{ij}(t);\quad z_i(0) = 0;\quad (i = 1, 2, ..., m;$$
$$t = 0, 1, ..., T-1) \tag{1}$$

$$y_j(t+1) = y_j(t) + \sum_{i=1}^{m} u_{ij}(t) - b_j(t);\ y_j(0) = 0;\quad (j = 1, 2, ..., n;$$
$$t = 0, 1, ... T-1) \tag{2}$$

$$0 \leqslant X_i(t) \leqslant a_i(t);\quad (i = 1, 2, ..., m;\quad t = 0, 1, ..., T-1) \tag{3}$$

$$0 \leqslant u_{ij}(t) \leqslant \delta_{ij}(t);\ (i=1, 2, ..., m;\ j=1, 2, ..., n;\ t=0, 1, ..., T-1) \tag{4}$$

$$0 \leqslant z_i(t) \leqslant \alpha_i(t);\quad (i = 1, 2, ..., m;\quad t = 1, 2, ..., T) \tag{5}$$

$$0 \leqslant y_j(t) \leqslant \beta_j(t),\quad (j = 1, 2, ..., n;\quad t = 1, 2, ..., T) \tag{6}$$

$$z_i(T) \geqslant 0;\quad (i = 1, 2, ..., m) \tag{7}$$

$$y_j(T) \geqslant 0;\quad (j = 1, 2, ..., n) \tag{8}$$

$$\min J_T(X, u) = \min_{X,u} \left[\sum_{t=0}^{T-1} \sum_{i=1}^{m} p_i(t) \cdot X_i(t) + \sum_{t=0}^{T-1} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(t) \cdot u_{ij}(t) + \right.$$
$$\left. + \sum_{t=1}^{T-1} \sum_{i=1}^{m} h_i(t) \cdot z_i(t) + \sum_{t=1}^{T-1} \sum_{j=1}^{n} q_j(t) \cdot y_j(t) \right] \tag{9}$$

The model (1) — (9) is a linear dynamic model which contains $m + n$ state variables and $n + mn$ decision variables.

Equations (1) and (2) describe the dynamics of the system, relations (3) and (4) represent the constraints upon the controls, relations (5) and (6) represent the constraints on the state of the system, and relations (7) and (8) represent the final conditions for the system.

We outline the fact that numerous classes of econo mic problems can be modelled by a model of the type (1) — (9).

Noting by:

$x_{ij}(\tau, \sigma, t)$ — the quantity which will be produced at production point i in period τ and transported in period σ to consumer j, in order to cover his demand in period $t (i = 1, 2, ..., m;\ j = 1, 2, ..., n;\ \tau = 0, 1, ..., T-1;\ t = \tau, \tau+1, ..., T-1;\ \sigma = \tau, \tau+1, ..., t)$ the model (1) — (9) becomes:

$$\sum_{j=1}^{n}\sum_{t=\tau}^{T-1}\sum_{\sigma=\tau}^{t} x_{ij}(\tau, \sigma, t) = X_i(\tau) \leqslant a_i(\tau);\quad (i = 1, 2, ..., m;$$

$$\tau = 0,1, ..., T-1) \tag{10}$$

$$\sum_{i=1}^{m}\sum_{\tau=0}^{t}\sum_{\sigma=\tau}^{t} x_{ij}(\tau, \sigma, t) = b_j(t);\quad (j = 1, 2, ..., n;$$

$$t = 0, 1, ..., T-1) \tag{11}$$

$$\sum_{j=1}^{n}\sum_{\tau=0}^{t-1}\sum_{k=t+1}^{T-1}\sum_{\sigma=t+1}^{k} x_{ij}(\tau, \sigma, k) = z_i(t) \leqslant \alpha_i(t);$$

$$(i = 1, 2, ..., m;\quad t = 1, 2, ..., T) \tag{12}$$

$$\sum_{i=1}^{m}\sum_{\tau=0}^{t-1}\sum_{k=t+1}^{T-1}\sum_{\sigma=\tau}^{t-1} x_{ij}(\tau, \sigma, k) = y_j(t) \leqslant \beta_j(t);$$

$$(j = 1, 2, ..., n;\quad t = 1, 2. ..., T) \tag{13}$$

$$\sum_{\tau=0}^{t}\sum_{k=t}^{T-1} x_{ij}(\tau, t, k) = u_{ij}(t) \leqslant \delta_{ij}(t);$$

$$(i = 1, 2, ..., m;\quad j = 1, 2, ..., n;\quad t = 0, 1, ..., T-1) \tag{14}$$

$$x_{ij}(\tau, \sigma, t) \geqslant 0;\quad (i = 1, 2, ..., m;\quad j = 1, 2, ..., n;\quad \tau = 0, 1, ..., T-1;$$

$$t = \tau, \tau+1, ..., T-1;\quad \sigma = \tau, \tau+1, ..., t) \tag{15}$$

$$min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\tau=0}^{T-1}\sum_{t=\tau}^{T-1}\sum_{\sigma=\tau}^{t}\Bigg[p_i(\tau) + \sum_{k=\tau+1}^{\sigma} h_i(k) + c_{ij}(\sigma) +$$

$$+ \sum_{k=\sigma+1}^{t} q_j(k)\Bigg] \cdot x_{ij}(\tau, \sigma, t). \tag{16}$$

In the case in which the storage and transportation capacities are unlimited, that is

$$\alpha_i(t) = \infty\,; \quad \beta_j(t) = \infty\,; \quad \delta_{ij}(t) = \infty$$

$$(i = 1, 2, ..., m;\ \ j = 1, 2, ..., n;\ \ t = 0, 1, ...\ T) \tag{17}$$

the model (10) — (16) is equivalent to the following model:

$$\sum_{j=1}^{n}\sum_{t=\tau}^{T-1} x_{ij}(\tau, t) = X_i(\tau) \leqslant a_i(\tau)\,; \quad (i=1, 2, ..., m;\ \tau=0, 1, ..., T-1) \tag{18}$$

$$\sum_{i=1}^{m}\sum_{\tau=0}^{t} x_{ij}(\tau, t) = b_j(t)\,; \quad (j = 1, 2, ..., n;\ \ t = 0, 1, ...\ T-1) \tag{19}$$

$$x_{ij}(\tau, t) \geqslant 0\,; \quad (i = 1, 2, ..., m;\ \ j = 1, 2, ..., n;$$

$$\tau = 0, 1, ..., T-1\,; \quad t = \tau, \tau+1, ..., T-1) \tag{20}$$

$$min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\tau=0}^{T-1}\sum_{t=\tau}^{T-1} \gamma_{ij}(\tau, t) \cdot x_{ij}(\tau, t) \tag{21}$$

where

$$x_{ij}(\tau, t) = \sum_{\sigma=\tau}^{t} x_{ij}(\tau, \sigma, t) \tag{22}$$

$$\gamma_{ij}(\tau, t) = p_i(\tau) + \min_{\tau \leqslant \sigma \leqslant t}\left[\sum_{k=\tau+1}^{\sigma} h_i(k) + c_{ij}(\sigma) + \sum_{k=\sigma+1}^{t} q_j(k)\right];$$

$$(i = 1, 2, ..., m;\ \ j = 1, 2, ..., n;\ \ \tau = 0, 1, ..., T-1;$$

$$t = \tau, \tau+1, ..., T-1). \tag{23}$$

The variable $x_{ij}(\tau, t)$ represents the quantity which has to be produced production point i in period τ and consumed by consumer j in period t ($t \geqslant \tau$).

The quantity $\gamma_{ij}(\tau, t)$ represents the minimum cost corresponding to an unit of the product produced at production point i in period τ and consumed by consumer j in period t. We have, for instance:

$$\gamma_{ij}(t, t) = p_i(t) + c_{ij}(t).$$

$$\gamma_{ij}(t-1, t) = p_i(t-1) + min\,[h_i(t) + c_{ij}(t)\,;\ \ c_{ij}(t-1) + q_j(t)]$$

$$\gamma_{ij}(t-2, t) = p_i(t-2) + min\,[h_i(t-1) + h_i(t) + c_{ij}(t)\,;\ \ h_i(t-1) +$$

$$+ c_{ij}(t-1) + q_j(t)\,;\ \ c_{ij}(t-2) + q_j(t-1) + q_j(t)].$$

a.s.o.

Noting:

$$x_{ij}(\tau, t) = x_{i+m\tau,\ j+nt}$$

$$a_i(\tau) = a_{i+m\tau}; \quad b_j(t) = b_{j+nt}; \tag{24}$$

$$\gamma_{ij}(\tau, t) = \gamma_{i+m\tau,\ j+nt};$$

$$(i = 1, 2, ..., m; \quad j = 1, 2, ..., n; \quad \tau = 0, 1, ..., T-1;$$

$$t = \tau, \tau + 1, ..., T-1)$$

the model (18) — (21) becomes:

$$\begin{cases} \sum\limits_{j=\left[\frac{i-1}{m}\right]n+1}^{nT} x_{ij} \leqslant a_i; \quad (i = 1, 2, ..., mT) & (25) \\ \sum\limits_{i=1}^{\left[\frac{j-1}{n}\right]m+m} x_{ij} = b_j; \quad (j = 1, 2, ..., nT) & (26) \\ x_{ij} \geqslant 0; \left(i = 1, 2, ..., mT; \quad j = \left[\frac{i-1}{m}\right]n + 1, ..., nT\right) & (27) \\ min \sum\limits_{i=1}^{mT} \sum\limits_{j=\left[\frac{i-1}{m}\right]n+1}^{nT} \gamma_{ij} x_{ij} & (28) \end{cases}$$

where $[k]$ — represents the integer part of number k.

The model (25) — (28) represents the model of an ordinary transportation problem, in which the routes of the form (i, j) with $j \leqslant \left[\frac{i-1}{m}\right]n$ are blocked.

Noting

$$\bar{\gamma}_{ij} = \begin{cases} \gamma_{ij}, & j \geqslant \left[\frac{i-1}{m}\right]n + 1 \\ \infty, & j < \left[\frac{i-1}{m}\right]n + 1 \end{cases}$$

$$(i = 1, 2, ..., mT; \quad j = 1, 2, ..., nT)$$

the model (25) — (28) becomes;

$$\sum_{j=1}^{nT} x_{ij} \leqslant a_i; \quad (i = 1, 2, ..., mT) \tag{25'}$$

$$\sum_{i=1}^{mT} x_{ij} = b_j; \quad (j = 1, 2, ..., nT) \tag{26'}$$

$$x_{ij} \geqslant 0; \quad (i = 1, 2, ..., mT; \quad j = 1, 2, ..., nT) \tag{27'}$$

$$min \sum_{i=1}^{mT} \sum_{j=1}^{nT} \overline{\gamma}_{ij} \cdot x_{ij}. \tag{28'}$$

In order that the model (1) — (9) with $\alpha_i(t) = \beta_j(t) = \delta_{ij}(t) = \infty$ for $i = 1, 2, ..., m$; $j = 1, 2, ..., n$; $t = 0, 1, ..., T - 1$ had a solution, it is necessary and sufficient that [7]:

$$\sum_{t=0}^{\tau} \sum_{i=1}^{m} a_i(t) \geqslant \sum_{t=0}^{\tau} \sum_{j=1}^{n} b_j(t); \quad (\tau = 0, 1, ..., T - 1).$$

In order to solve the model (25') — (28'), one can apply any of the usual algorithms for the solution of the transportation problem, for instance the *method of compared differences* proposed by the author in 1961 and described in [1], [2], [3], [4].

Since problem (25) — (28), or (25') — (28') has many blocked routes, its matrix having, in fact, a triangular form , (Fig. 1), it can be solved by means of a decomposition technique based on the method of compared differences.

Further, we make a brief exposition of the method of compared differences.

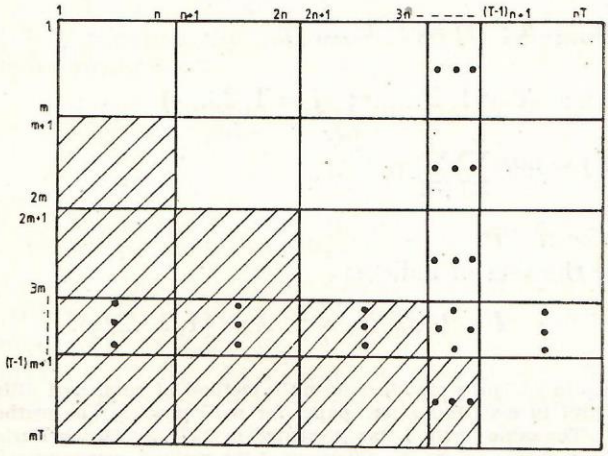

Fig. 1

3. A BRIEF EXPOSITION OF THE METHOD OF COMPARED DIFFERENCES*

We consider the transportation problem:

$$(P): \begin{cases} \sum_{j=1}^{q} x_{ij} \leqslant a_i; \quad (i = 1, 2, ..., r) & (29) \\ \sum_{i=1}^{r} x_{ij} = b_j; \quad (j = 1, 2, ..., q) & (30) \\ x_{ij} \geqslant 0; \quad (i = 1, 2, ..., r; \quad j = 1, 2, ..., q) & (31) \\ H = min \sum_{i=1}^{r} \sum_{j=1}^{q} \gamma_{ij} x_{ij} & (32) \end{cases}$$

where $a_i (i = 1, 2, ..., r)$ and $b_j (j = 1, 2, ..., q)$ are positive integer numbers, and $\sum_{i=1}^{r} a_i \geqslant \sum_{j=1}^{q} b_j$.

Definition 1. Let us consider the vector $a^k = (a_1^k, a_2^k, ..., a_r^k)$ with $a_i^k \geqslant 0$ and

$$\sum_{i=1}^{r} a_i^k = \sum_{j=1}^{q} b_j \cdot$$

An optimal solution $X(a^k) = \|x_{ij}(a^k)\|_{r \times q}$ of the problem:

$$P(a^k): \begin{cases} \sum_{j=1}^{q} x_{ij} = a_i^k; \quad (i = 1, 2, ..., r) & (29') \\ \sum_{i=1}^{r} x_{ij} = b_j; \quad (j = 1, 2, ..., q) & (30') \\ x_{ij} \geqslant 0; \quad (i = 1, 2, ..., r; \quad j = 1, 2, ...q) & (31') \\ H(a^k) = min \sum_{i=1}^{r} \sum_{j=1}^{q} \gamma_{ij} \cdot x_{ij} & (32') \end{cases}$$

is called a *choice* for P.

Let us consider the sets of indices:

$$I = \{1, 2, ..., r\}; \quad J = \{1, 2, ..., q\}.$$

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*) We present in a slightly modified form the method of compared differences proposed by the author in 1961 in his Graduation paper for mathematics. The method was described also in [2], [3], [4]. The same method was presented in 1977 by Ulrich Derigs and Uwe Zimmermann [6]. These authors analyze the efficiency of the method, compared with other metho ds for the solution of the transportation problem.

Corresponding to the choice $X(a^k)$, we define the following partition of I:

$$I^\circ(a^k) = \{i : i \in I; \quad a_i^k = a_i\}$$
$$I^+(a^k) = \{i : i \in I; \quad a_i^k > a_i\} \tag{34}$$
$$I^-(a^k) = \{i : i \in I; \quad a_i^k < a_i\}$$

$I^+(a^k)$ represents the set of production points whose production capacity is oversolicited according to the choice $X(a^k)$, and $I^-(a^k)$ is the set of undersolicited production points.

Definition 2. The number:

$$mes\ X(a^k) = \sum_{i \in I^+(a^k)} [a_i^k - a_i] \tag{35}$$

is called the *measure of the choice* $X(a^k)$.

If $mes\ X(a^k) = 0$, then the choice $X(a^k)$ represents an optimal solution of problem P. The fact that $mes\ X(a^k) = 0$ is equivalent to the fact that $I^+(a^k) = \Phi$.

Definition 3. Corresponding to the choice $X(a^k)$, we define for any pair $(i_1, i_2) \in I \times I$ the following „cost":

$$\alpha_{i_1 i_2}(a^k) = \min_{j \in M_{i_1}(a^k)} [\gamma_{i_2 j} - \gamma_{i_1 j}] \tag{36}$$

where

$$M_{i_1}(a^k) = \{j : j \in J, (i_1, j) \in \Omega(a^k)\} \tag{37}$$

and

$$\Omega(a^k) = \{(i, j) : (i, j) \in I \times J; \quad x_{ij}(a^k) > 0\}. \tag{38}$$

In general,

$$\alpha_{i_1 i_2}(a^k) \neq \alpha_{i_2 i_1}(a^k).$$

Definition 4. We define the „distance" from the set $I^+(a^k)$ to the set $I^-(a^k)$ in the following way:

$$\Delta_{i_0}(a^k) = \min_{i_0 \in I^+(a^k)} \min_{i_p \in I^-(a^k)} \eta_{i_0 i_p}(a^k); \tag{39}$$

where

$$\eta_{i_0 i_p}(a^k) = min\ [\alpha_{i_0 i_p}(a^k); \min_{i_h i_l \in I^\circ(a_k)} (\alpha_{i_0 i_h}(a^k) + \varepsilon_{i_h i_l}(a^k) + \alpha_{i_l i_p})] \tag{40}$$

and $\varepsilon_{i_h i_l}(a^k)$ represents the minimum distance between the vertices i_h, i_l of the graph $G(a^k) = (I^\circ(a^k), \Gamma)$ generated by the set $I^\circ(a^k)$, in which the length of the arch (i_h, i_l) is $\alpha_{i_h i_l}(a^k)$.

The numbers $\varepsilon_{i_h i_l}(a^k)$ can be computed, for instance, by the aid of *B.* Roy's algorithm [8] or any other algorithm for the determination of minimal paths in an oriented graph (see, for instance, the algorithm presented in [3]).

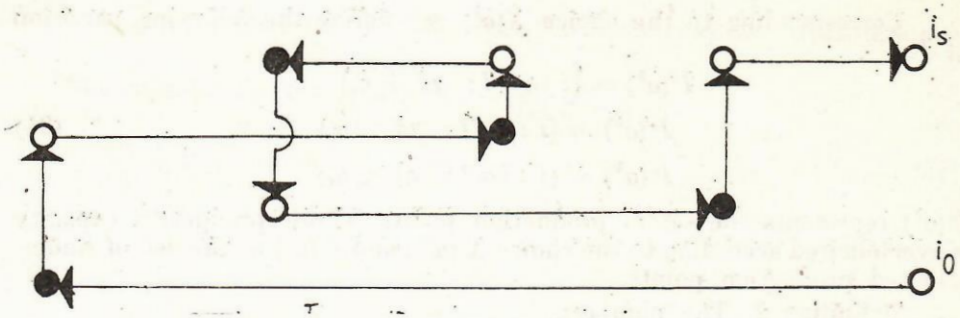

Fig. 2

Definition 5. Let $\Delta(a^k)$ be the "distance" from the set $I^+(a^k)$ to the set $I^-(a^k)$; if:

$$\Delta(a^k) = \sum_{l=1}^{s} [\gamma_{i_l j_{l-1}} - \gamma_{i_l\, j_{l-1}}] \tag{41}$$

then

$$\mathcal{L}(a^k) = \{(i_0, j_0),\ (i_1, j_0),\ (i_1, j_1),\ (i_2, j_1), ..., (i_{s-1}, j_{s-1}),\ (i_s, j_{s-1})\} \tag{42}$$

is called *connection path* between $I^+(a^k)$ and $I^-(a^k)$.

According to the construction, we have:

$$i_0 \in I^+(a^k);\quad i_l \in I^\circ(a^k) \text{ for } l = 1, 2, ..., s-1;\quad i_s \in I^-(a^k) \tag{43}$$

and

$$(i_l, j_l) \in \Omega\ (a^k) \quad \text{for } l = 0, 1, ..., s-1. \tag{44}$$

The path (42) is presented in fig. 2.

Definition 6. The *capacity of the connection path* (42) between $I^+(a^k)$ and $I^-(a^k)$ is:

$$c(\mathcal{L}(a^k)) = \min_{l=0,1,...,s-1} x_{i_l j_l}(a^k). \tag{45}$$

We note:

$$\mathcal{L}^\circ(a^k) = \{(i_0, j_0),\ (i_1, j_1), ..., (i_{s-1}, j_{s-1})\} \tag{46}$$

$$\mathcal{L}^1(a^k) = \{(i_1, j_0),\ (i_2, j_1) ..., (i_s,\ j_{s-1})\}. \tag{47}$$

Theorem. Let $\Delta(a^k)$ be the "distance" from the set $I^+(a^k)$ to the set $I^-(a^k)$ and $\mathcal{L}(a^k)$ the corresponding path, given by (42).

If

$$a_i^{k+1} = \begin{cases} a_i^k - v(a^k) & \text{for } i = i_o \\ a_i^k + v(a^k) & \text{for } i = i_s \\ a_i^k & \text{for } i \neq i_0 \text{ and } i \neq i_s \end{cases} \tag{48}$$

where

$$v(a^k) = min\,\{a_{i_o}^k - a_{i_o};\quad a_{i_s} - a_{i_s}^k;\quad c(\mathcal{L}(a^k))\} \tag{49}$$

and $c(\mathcal{L}(a^k))$ is the capacity of the path $\mathcal{L}(a^k)$, then the choice $X\ (a^{k+1})$ is defined by:

$$x_{ij}(a^{k+1}) = \begin{cases} x_{ij}(a^k) & \text{if } (i,j) \notin \mathcal{L}(a^k) \\ x_{ij}(a^k) - v(a^k) & \text{if } (i,j) \in \mathcal{L}^0(a^k) \\ x_{ij}(a^k) + v(a^k) & \text{if } (i,j) \in \mathcal{L}^1(a^k) \end{cases} \tag{50}$$

and

$$H(a^{k+1}) = H(a^k) + v(a^k) \cdot \Delta(a^k) \tag{51}$$

where $H\ (a^k)$ is the optimum value of the objective-function (32').

We also have:

$$mes\ X(a^{k+1}) = mes\ X(a^k) - v(a^k) < mes\ X(a^k). \tag{52}$$

4. THE ALGORITHM OF COMPARED DIFFERENCES

Based on the content of section 3, we present the algorithm of compared differences for the solution of transportation problem P given by relations (29) — (32).

Step 1. We compute the initial choice $X^\circ = \|x^\circ_{ij}\|_{r\times q}$. In order to do this, the numbers γ_{ij} are put in increasing order, for any j:

$$\gamma_{i_1 j} \leqslant \gamma_{i_2 j} \leqslant \ldots \leqslant \gamma_{i_r j}; \quad (j = 1, 2, \ldots, q).$$

We determine

$$x^\circ_{i_k j} = max\left\{0,\ min\left[b_j - \sum_{l=1}^{k-1} x_{i_l j};\ \ a_{i_k}\right]\right\} \tag{53}$$

By definition:

$$a^\circ_i = \sum_{j=1}^{q} x^\circ_{ij}; \quad (i = 1, 2, \ldots, r) \text{ and } X(a^\circ) = X^\circ; \tag{54}$$

we put $k := 0$ and pass to step 2.

Step 2. For $X\ (a^k)$ we identify the sets $I^\circ(a^k)$, $I^+(a^k)$ and $I^-(a^k)$ given by formulas (34).

If $mes\ X\ (a^k) = 0$, then the choice $X\ (a^k)$ represents the optimal solution of problem P and the computation stops.

If $mes\ X\ (a^k) > 0$, we pass to step 3.

Step 3. We compute the costs $\alpha_{i_1 i_2}(a^k)$ for $i_1 \in I^\circ\ (a^k) \cup I^+(a^k)$ and $i_2 \in I^\circ(a^k) \cup I^-(a^k)$ according to formula (36).

We compute the matrix

$$\varepsilon(a^k) = \|\varepsilon_{i_h i_l}(a^k)\|_{r(k)\times r(k)} \tag{55}$$

of minimum distances between the vertices of the graph $(I^{\circ}(a^k), \Gamma)$ generated by the set $I^{\circ}(a^k)$, where the length of the arch (i_1, i_2) is $\alpha_{i_1 i_2}(a^k)$ given by (36). We noted:

$$r(k) = card\ I^{\circ}(a^k). \tag{56}$$

Step 4. Let be $i_0 \in I^{+}(a^k)$, $i_s \in I^{-}(a^k)$ the indices realizing $\Delta(a^k)$, the minimum distance between $I^{+}(a^k)$ and $I^{-}(a^k)$, according to formula (39); we identify the connection path $\mathcal{L}(a^k)$ between $I^{+}(a^k)$ and $I^{-}(a^k)$ (according to definition 5).

We compute the new vector a^{k+1}, according to formulas (48) — (49) and the corresponding choice $X(a^{k+1})$, according to formulas (50).

We change k into $k+1$ and return to step 2.

Since a_i and b_j are integers and, according to the theorem of section 3, we have:

$$mes\ X(a^{k+1}) < mes\ X(a^k), \tag{57}$$

the algorithm will lead, in a finite number of steps, to a choice $X(a^p)$ such that:

$$mes\ X(a^p) = 0$$

and, therefore, to the solution of problem P.

Remarks

1. We have

$$H(a) = H(a^{\circ}) + \sum_{k=0}^{p-1} \varphi(a^k) \cdot \Delta(a^k) \tag{58}$$

where $H(a)$ represents the optimal value of the performance of problem P, p — *the* number of iterations necessary to reach the choice $X(a^p)$ with null measure.

2. Let $\Delta(a^k)$ be the distance between sets $I^{+}(a^k)$ and $I^{-}(a^k)$ corresponding to choose $X(a^k)$; we have

$$\Delta(a^{\circ}) \leqslant \Delta(a^1) \leqslant \ldots, \leqslant \Delta(a^p) \tag{59}$$

3. The application of compared diferences method is efficient, particularly in the case in which problem matrix is very rectangular, respectively

$$\frac{r}{q} < 1. \tag{60}$$

4. In the case in which problem (P) contains constraints for the capacities, that is constraints (31) become

$$0 \leqslant x_{ij} \leqslant \delta_{ij}; \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, q), \tag{31'}$$

then the algorithm has to be modified in the following way:

a) Let us consider, for every j, the quantities γ_{ij} put in increasing order:

$$\gamma_{i_1 j} \leqslant \gamma_{i_2 j} \leqslant \ldots, \leqslant \gamma_{i_r j}.$$

The initial choice X° (step 1) is computed in the following way:

$$x^0_{i_k j} = max\left\{0,\ min\left[\delta_{i_k j};\quad b_j - \sum_{l=1}^{k-1} x_{i_l j};\quad a_{i_k}\right]\right\}. \tag{53'}$$

b) The capacity of the connection path between $I^+(a^k)$ and $I^-(a^k)$ (Definition 6) is given by the formula:

$$c(\mathcal{L}(a^k)) = min\left\{\min_{l=0,1,\ldots,s-1} x_{i_l j_l}(a^k),\ \min_{l=1,2,\ldots,s} \delta_{i_l j_{l-1}}\right\}. \tag{45'}$$

5. THE SOLUTION OF THE DYNAMIC PRODUCTION-TRANSPORTATION PROBLEM BY THE AID OF THE ALGORITHM OF COMPARED DIFFERENCES.

Let us consider the dynamic production-transportation problem without constraints concerning the storage and transportation capacities. As we have shown in section 1, this is equivalent to problem (25) — (28).

Let $\{P_t\}$ $t = 0, 1, ..., T-1$ be the following family of transportation problems:

$$(P_t)\ :\ \begin{cases} \sum\limits_{j=nt+1}^{(t+1)n} x_{ij} \leqslant a_i; \quad (i = 1, 2, ..., (t+1)\,m) & (61) \\ \sum\limits_{i=1}^{(t+1)m} x_{ij} = b_j; \quad (j = nt+1, ..., (t+1)\,n) & (62) \\ x_{ij} \geqslant 0; \quad (i = 1, 2, ..., (t+1)m;\ \ j = nt+1, ..., (t+1)n) & (63) \\ min \sum\limits_{i=1}^{(t+1)m} \sum\limits_{j=nt+1}^{(t+1)n} \gamma_{ij} \cdot x_{ij}; & (64) \end{cases}$$

and $\overline{X}^0_t$ — the optimal solution of problem $P_t (t = 0, 1, ..., T-1)$. This can be obtained by means of the algorithm of compared differences, or any other algorithm for the solution of the transportation problem.

Let us consider $X^\circ = \| x^0_{ij} \|_{mT \times nT}$, where:

$$x^0_{ij} = \begin{cases} \bar{x}^0_{ij} \text{ if } j \geqslant \left[\dfrac{i-1}{m}\right] n + 1 \\ 0 \text{ if } j < \left[\dfrac{i-1}{m}\right] n + 1 \end{cases}; \quad (i = 1, 2, ..., Tm) \tag{65}$$

We define

$$a^0_i = \sum_{j=\left[\frac{i-1}{m}\right]n+1}^{nT} x^0_{ij}; \quad (i = 1, 2, ..., mT) \tag{66}$$

and

$$X(a^\circ) = X^\circ \tag{67}$$

which represents the initial choice for problem (25) — (28).

We have:

$$a_i^0 \leqslant \left(T - \left[\frac{i-1}{n}\right]\right) a_i; \quad (i = 1, 2, ..., mT). \tag{68}$$

Let us consider the sets of indices:

$$I = \{1, 2, ..., mT\}; \quad J = \{1, 2, ..., nT\}. \tag{69}$$

In order to find the solution of problem (25) — (28), we apply the algorithm of compared differences, described in the preceeding section, with the single modification that the quantities $\alpha_{i_1 i_2}(a^k)$ are defined by the formula:

$$\alpha_{i_1 i_2}(a^k) = \min_{j \in M_{i_1 i_2}(a^k)} [\gamma_{i_2 j} - \gamma_{i_1 j}] \tag{36'}$$

where:

$$M_{i_1 i_2}(a^k) = \left\{j : j \in J; \quad j \geqslant \left[\frac{i_2 - 1}{m}\right] n + 1; \quad (i_1, j) \in \Omega(a^k)\right\} \tag{37'}$$

and $\Omega\ (a^k)$ is given by (38).

As concerns the connection path (42), besides relations (43), (44), condition

$$j_l \geqslant \left[\frac{i_{l+1} - 1}{m}\right] n + 1; \quad (l = 0, 1, ..., s-1) \tag{70}$$

is also satisfied.

Let $X(a^p)$ be a choice for problem (25) — (28) with:

$$mes\ X\ (a^p) = 0.$$

It follows that $X\ (a^p)$ is an optimal solution of problem (25) — (28). In this case, the corresponding optimal solution of problem (1) — (9), with conditions (17) satisfied, is the following:

$$X_i(t) = \sum_{j=nt+1}^{nT} x_{mt+i,j}; \quad (i = 1, 2, ..., m; \quad t = 0, 1, ..., T-1) \tag{71}$$

$$u_{ij}(t) = \sum_{(\tau,\sigma) \in \Lambda_{ij}(t)} x_{i+m\tau, j+n\sigma}; \quad (i = 1, 2, ..., m; \quad j = 1, 2, ..., n;$$

$$t = 0, 1, ..., T-1) \tag{72}$$

where

$$\Lambda_{ij}(t) = \{(\tau, \sigma) : 0 \leqslant \tau \leqslant t; \quad t \leqslant \sigma \leqslant T-1; \quad \min_{\tau \leqslant k \leqslant \sigma} \varphi_{ij}(\tau, k, \sigma) =$$

$$= \varphi_{ij}(\tau, t, \sigma)\} \tag{73}$$

whith

$$\varphi_{ij}(\tau, k, \sigma) = \sum_{l=\tau+1}^{k} h_i(l) + c_{ij}(k) + \sum_{l=k+1}^{\sigma} q_j(l). \tag{74}$$

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UN ALGORITHME POUR LA RÉSOLUTION DU PROBLÈME DYNAMIQUE DE PRODUCTION-TRANSPORT

Dans cet article on présente le problème dynamique général de production-transport; ensuite, ce problème est transformé en un problème de programmation linéaire d'un type spécial. On montre que, dans la situation où il n'y a pas de contraintes concernant les capacités démmagasinage chez les producteurs ou les consommateurs et o'il n'existe pas de routes de transport à capacité limitée, le problème dynamique peut êntre réduit à un problème ordinaire de type „transport“. On présente l'algorithme des différences comparées pour la résolution des problèmes de transport. Ensuite, cet algorithme est appliqué à la résolution du probleme dynamique production-transport.

АЛГОРИТМ ДЛЯ РЕШЕНИЯ ДИНАМИЧЕСКОЙ ПРОИЗВОДСТВЕННОЙ-ТРАНСПОРТНОЙ ЗАДАЧИ

В работе излагается общая динамическая производетвенная — транспортная задача; потом, эта задача сводится к задаче линейного программирования специального типа. Показывается что, в случае когда ограничения на способности складывания у производителя или у потребителя отсутствуюм, а ограничений на транспортные способности маршрут также не имеются, динамическая задача сводится к обыкновенной задаче транспортного типа. Излагается алгоритм сравнённых разностей для решения транспортных задач, а также способ его применения к решению динамической производственой-транспортной задачи.

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