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Mathematical Programming Real Variable Models for Integer and Mixed-Integer Problems

In the paper the authors propound, for some problems with discrete or mixed-integer variables, alternative real variable models, based upon function $\delta(x) = x/(\alpha + x)$, $\alpha > 0$ *or upon some of its compositions, as for instance* $\sigma(x) = \sum_i \delta_i(x_i)$, $\Pi(x) = \prod_i \delta_i(x_i)$ *a. s. o.*

It is shown, for instance, that there exists a vector $\alpha > 0$, *such that the solution to problem* $\min \sigma(x, \alpha) = \sum_{i \in I} x_i/(\alpha_i + x_i)$, $Ax = b$, $x \geqslant 0$, *is a solution to problem* $\min |x|^{+}$, $Ax = b$, $x \geqslant 0$ where *the cardinal of x, i.e. the number of strictly positive components of x, is denoted by* $|x|^{+}$, *thus offering another alternantive for Cabot's procedure (1975) of determining in real an admissible solution to a generalized lattice point problem.*

Continuous models are also propounded for problems with disjunctive constraints , generalizing the Raghavachari results (1967), as well as models for the problem of minimal covering and for the minimal set partitioning or the set packaging problem.

1. A CLASS OF FUNCTIONS WITH QUASI-DISCRETE VALUES

1.1. A function with quasi-boolean values

Let function

$$\delta(x) = x/(\alpha + x), \quad \alpha \neq 0 \tag{1}$$

be defined on $R \setminus \{-\alpha\}$ with values in R. This function has "nearly everywhere" in a generalized sense", in R "values nearly equal to one". We say that a pro-

perty of some elements of a set takes place nearly everywhere in a generalized sense, if the ratio between the number of elements (or their measure, respectively) which do not posses the property and the number of elements (or their measure, respectively) which have the property, can be made smaller than a given number, positive and arbitrarily small. We shall say that the values of a function are nearly discrete if, nearly everywhere, they can be made to differ from an integer with a no matter haw little quantity. We shall study, in what follows, for rendering easy the exposition, the properties in $R_+ = \{ x \mid x \geqslant 0\}$, for $\alpha > 0$. It results immediately that the function $\delta(x)$ has the property of having almost everywhere values almost equal to one. Indeed, for $\alpha > 0$

$$\delta(x,\ x \geqslant r) \geqslant 1 - \alpha/(r + \alpha) \tag{2}$$

and always, whichever would be $r > 0$, we can select α so that $\delta(x)$ be no matter how near to 1, nearly everywhere in R_+.

As $\delta(0) = 0$, and, for $0 < \varepsilon < 1$,

$$\delta[x|0 \leqslant x \leqslant \alpha\varepsilon/(1 - \varepsilon)] \leqslant \varepsilon \tag{3}$$

and

$$\delta[x|x \geqslant \alpha(1 - \varepsilon) / \varepsilon] \geqslant 1 - \varepsilon \tag{4}$$

we can say that function δ has nearly everywhere quasi-boolean values. We can consider function δ as a membership function, in the sense of fuzzy sets, wherein the origin has a zero membership degrees, the points in the vecinity of origin have a ε membership degree, and for the great values of the x, the membership degree grows and asimptotically tends to 1.

An immediate generalization is the function

$$\delta(x,\ N,\ p,\ q,\ \alpha) = [px/(qx + \alpha)] \cdot N;\quad p,\ q,\ \alpha > 0. \tag{5}$$

where N is a natural number, which , for $x \gg \alpha$, has values close to $N\ p/q$ and hence, for $p = q$, close to N. Function δ is an always increasing function; it is continuous in $R \setminus \{ - \alpha/q\}$, convex for $x < - \alpha/q$ and concave for $x > - \alpha/q$, and posseses derivatives of any order in the domain of definition.

Passing over from an one-dimensional space to R^n, we will define in the followings some elementary combinations with function δ.

1.2. Function σ with nearly integer values between zero and n.

We define function σ on R^n, for $x_i \neq - \alpha_i$, $i = 1, ..., n$, equal to

$$\sigma(x) = \sum_{i=1}^{n} x_i/(\alpha_i + x_i),\ \alpha_i > 0,\ i = 1, ..., n, \tag{6}$$

which is a function with separate variables, with values in R. As before, especially the properties of the function on R_+^n will be of interest. As

$$\partial\sigma/\partial x_j = \alpha_j/(\alpha_j + x_j)^2 \tag{7}$$

$$\partial^2\sigma/\partial x_j^2 = - 2\alpha_j/(\alpha_j + x_j)^3, \partial^2\sigma/\partial_{x_i}\partial_{x_j} = 0,\ i \neq j \tag{8}$$

it results that function σ is increasing as against each variable, and concave on R_+^n. In fact, function σ is strictly increasing on $D = \{x | x \in R_+^n\, x_i \neq -\alpha_i$ $i = 1, ..., n\}$ in the sense that $x^1 < x^2$, where we mean by $x^1 < x^2$: $x_j^1 \leqslant x_j^2$, $x^1 \neq x^2$, implies $\sigma(x^1, \alpha^\circ) < \sigma(\alpha^2, \alpha^\circ)$.

Taking (4) into account, we have

$$k > \sigma[x,\ \alpha \mid x \in R_+^k, x_i \geqslant (\alpha_i/\varepsilon_i) - \alpha_i] \geqslant k\left(1 - \sum_{i=1}^{k} \varepsilon_i\right), \tag{9}$$

$$\alpha_i > 0,\ \varepsilon_i > 0,$$

hence we may regard function $\mu = \sigma/k$ as a ε membership function to R_+^k we could also say that it is a „ε characteristic" function.

If we denote by e_x the vector having $(e_x)_i = 1$, if $x_i > 0$ and $(e_x)_i = 0$ if $x_i = 0$, then we have

$$|x|^+ > \sigma[x, \alpha \mid x \in R_+^n,\ 0 < \alpha_i \leqslant \alpha_0 \varepsilon_i,\ x_i \neq 0 \Rightarrow x_i \geqslant$$

$$\geqslant \alpha_0(1 - \varepsilon_i)] \geqslant |x|^+(1 - \varepsilon^T e_x) \tag{10}$$

where ε_i denotes the components of vector ε and $|x|^+$ denotes the number of strictly positive components of x, i.e. the cardinal of x.

Let α^1, α^2 be two parameter vectors. Then we have

$$\alpha^1 > \alpha^2,\ x^\circ \neq 0 \Rightarrow \sigma(x^\circ,\ \alpha^1) < \sigma(x^\circ,\ \alpha^2) \tag{11}$$

$$\lim_{\alpha \searrow 0} \sigma(x^\circ, \alpha) = |x^\circ|^+. \tag{12}$$

1.3. ε — Characteristic functions of fractional type

Let

$$f(x) = \left(\sum_{j=1}^{m} x_j\right) \Big/ \left(\alpha_0 + \sum_{j=1}^{m} x_j\right),\ \alpha_0 > 0 \tag{13}$$

be the fractional function. For this function we have $f(0,\ \alpha_0) = 0$, but the function f is ε equal to 1, if at least one of the x_j is great enough as against α_0, or if $\sum x_j$ is great enough as against α_0. We have

$$1 > f(x \mid x \in R_+^m,\ \sum_{j=1}^{m} x_j \geqslant (\alpha_0/\varepsilon) - \alpha_0) \geqslant 1 - \varepsilon \tag{14}$$

$$\partial f/\partial x_j = \alpha_0 \Big/ \left(\alpha_0 + \sum_{i=1}^{m} x_i\right)^2. \tag{15}$$

Function (13) is strictly increasing on R_+^m and as

$$\partial^2 f/\partial x_j \partial x_k = -2\alpha_0 \Big/ \left(\alpha_0 + \sum_{j=1}^{m} x_j\right)^3, \tag{16}$$

it results that the hessian is negative definite in each point on R_+^m, hence $f(x,\ \alpha_0)$ is concave on R_+^m.

Using function (13), we can define on R^{nm} the function

$$F(x) = \sum_{i=1}^{n} f_{i.} = \sum_{i=1}^{n} \left[\sum_{j=1}^{m} x_{ij} / \left(\alpha_i + \sum_{j=1}^{m} x_{ij} \right) \right] \tag{17}$$

where

$$f_{i.} = \left(\sum_{j=1}^{m} x_{ij} \right) \Big/ \left(\alpha_i + \sum_{j=1}^{m} x_{ij} \right) \tag{18}$$

As

$$\partial F / \partial x_{ij} = \partial f_i / \partial x_{ij} = \alpha_i \left(\alpha_i + \sum_{j=1}^{m} x_{ij} \right)^2 \tag{19}$$

it results that $F(x)$ is an increasing function as against each component. As regards the hessian $\nabla^2 F(x)$, it is a matrix of the mn order, having on the diagonal n matrices of order m, whose elements are

$$\partial^2 f_i / \partial x_{ij} \partial x_{ik} = -2\alpha_i / \left(\alpha_i + \sum_{j=1}^{m} x_{ij} \right)^3, \quad \text{for} \quad \begin{matrix} i = 1, \ldots, n \\ j, k \in \{1, \ldots, m\} \end{matrix}$$

and all the other elements equal to zero.

1.4. The Product Function $\pi(x)$.

Function:

$$\pi(x) = \prod_{i=1}^{n} [x_i / (\alpha_i + x_i)], \quad \alpha_i > 0 \tag{20}$$

is an explicitely quasi-concave function on R_+^m. We have:

$$\pi(x \mid x \in R_+^m, \ m < n) = 0 \tag{21}$$

$$1 > \pi[x \mid x \in R_+^n, \ x_i \geqslant (\alpha_i / \varepsilon) - \alpha_i] \geqslant (1 - \varepsilon)^n \tag{22}$$

and a direct calculation shows that function $\Pi(x)$ is also strictly increasing as against each variable. The same properties remain valid also for:

$$\Pi_{\nu_i}(x) = \prod_{i=1}^{n} [x_i / (\alpha_i + x_i)]^{\nu_i}, \quad \alpha_i > 0, \ \nu_i > 0. \tag{23}$$

Function $\Pi_{\nu_i}(x)$ is an explicitely quasi-concave function on R_+^n[8], and we have:

$$1 > \pi[x \mid x \in R_+^n, \ x_i \geqslant (\alpha_i / \varepsilon_i) - \alpha_i] \geqslant \sum_{i=1}^{n} (1 - \varepsilon_i)^{\nu_i}. \tag{24}$$

1.5. Function $E(x)$. We define:

$$E(x) = exp \left(\sum_{i=1}^{n} \delta i \right) = \prod_{i=1}^{n} exp \ (\delta i). \tag{25}$$

As for $x_i > -\alpha_i/2$, $\exp(\delta i)$ is concave, it results [8] that $E(x) = \prod_{i=1}^{n} \exp(\delta i)$ is, for $x_i > -\alpha_i/2$, explicitely quasi-concave.

2. REAL MATHEMATICAL PROGRAMMING MODELS FOR INTEGER AND MIXED -INTEGER PROBLEMS

2.1. Let LCCM be the Linear Constrained Cardinal Minimization Problem:

$$min\ |x|^+, \text{ subject to } Ax = b,\ x \geqslant 0,$$

where we denote by $|x|^+$ the cardinal of x, i.e. the number of the strictly positive components of x, x non-negative and subject to linear constraints. We suppose in what follows, that the admissibility domain of LCCM problem is bounded, and let x_c^* be the solution to LCCM problem.

We have the following:

Theorem 2.1.1. There exists a vector $\alpha^T = (\alpha_1, ..., \alpha_n)$, $\alpha_i > 0$, so that the solution to problem $P\sigma$: $min\ \sigma(x) = \sum_{i=1}^{n} x_i/(\alpha_i + x_i)$, $Ax = b$, $x \geqslant 0$ is a solution to LCCM problem.

Proof

Let be $S = \{x \mid Ax = b,\ x \geqslant 0\}$. For any $x \in S$ we have $x = \sum_{i=1}^{k} \lambda_i x_v^i$, $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geqslant 0$ where x_v^i are the verteces of the polyhedron of the feasible solutions, and hence, as $x_v^i \geqslant 0$, we have

$$|x|^+ = \left|\sum_{i=1}^{k} \lambda_i x_v^i\right|^+ \geqslant \max_i\ |x_v^i|^+$$

i.e. the solution to LCCM problem is not an interior point of S. There can be several solutions, all vertices and even an infinity in the case when we have as solutions several edges of the polyhedron of solutions.

In any way, a solution to LCCM problem is a vertex.

Let $\mu_0 = \min_{i,j}\{x_{jv}^i > 0,\ \forall\ x_v^i\}$ and x_σ^* be the solution to problem $P\sigma$ with $\alpha_i = \mu_0 \varepsilon_i$. According to (10), we have

$$\sigma(x_\sigma^*) \geqslant |x_\sigma^*|^+ [1 - \varepsilon^T e_{x_\sigma^*}]. \tag{26}$$

If $|x_\sigma^*|^+ > |x_c^*|^+$, then, as $|x_c^*|^+ > \sigma\ (x_c^*)$, and taking ε sufficiently small, it results

$$\sigma(x_\sigma^*) > \sigma(x_c^*) \tag{27}$$

i.e. x_σ^* is not a point of minimum to $P\sigma$ problem. If $|x_\sigma^*|^+ < |x_c^v|^+$, then x_c^* is not a solution to LCCM problem.

Note 1. If LCCM problem has a unique solution, then both problems are equivalent.

Note 2. Let us assume that $x_\sigma^* > 0$ and let $\mu_\sigma^* > 0$ be the value of the smallest component, differing from zero, of x_σ^*. Then x_σ^* is a solution to minimum μ cardinal perturbed problem, (LCCMμ) $min\ |x|^+$, with constraints $Ax = b - \mu_\sigma^* A e_{x_\sigma^*}$, $x \geqslant 0$.

Indeed, the problem $min\ |x|^+$ with conditions $Ax = b - \mu^*_\sigma A e_{x^*_\sigma}$ is equivalent to problem $min\ |x|^+$, with conditions $Ax = b,\ x \geqslant \mu^*_\sigma e_{x^*_\sigma}$, which has $|x^*_\sigma|^+$ as solution.

Theorem 2.1.2. (sufficient condition)
A solution x^*_σ of problem $(P\sigma)$ is a solution to LCCM problem, if

$$|x^*_\sigma|^+ - \sigma(x^*_\sigma) < 1. \tag{28}$$

Let x^*_c be a solution to LCCM problem. We have the following sequence of inequalities:

$$|x^*_\sigma|^+ \geqslant |x^*_c|^+ > \sigma(x^*_c) \geqslant \sigma(x^*_\sigma). \tag{29}$$

If $|x^*_\sigma|^+ - \sigma(x^*_\sigma) < 1$, inequalities (29) imply $|x^*_\sigma|^+ = |x^*_c|^+$.

It does not necessarily result that if $|x^*_\sigma|^+ = |x^*_c|^+$, then $|x^*_\sigma|^+ - \sigma(x^*_\sigma) < 1$ or, the same thing, that if $|x^*_\sigma|^+ - \sigma(x^*_\sigma) > 1$ then $|x^*_\sigma|^+ \neq |x^*_\sigma|^+$.

2.2. An Algorithm for Solving LCCM Problem. Let at sequence k be vector α^k, and $x^*_{\sigma s}$ a vertex with the least number of components, known so far.

1. Find out

$x^*_{\sigma(\alpha^k)}$ such that $\sigma(x^*_{\sigma(\alpha^k)}) = \min\limits_{x \in S} \sigma(x,\ \alpha^k),\ S = \{x \mid Ax = b,\ x \geqslant 0\}$

2. Optimality test

If $|x^*_{\sigma(\alpha^k)}|^+ - \sigma(x^*_{\sigma(\alpha^k)},\ \alpha^k) < 1$, then $x^*_{\sigma(\alpha^k)}$ is a solution to LCCM problem, STOP.

3. If $|x^*_{\sigma(\alpha^k)}|^+ \leqslant |x^*_{\sigma s}|^+$ then $x^*_{\sigma s} := x^*_{\sigma(\alpha^k)}$.

The point $x^*_{\sigma s}$ is the solution to LCCMμ problem: $min\ |x|^+$, $Ax = b - \mu^*_{\sigma s}\ A e_{\sigma s}$.

If LCCMμ problem sufficiently approximates the LCCM problem, stop.

4. $\alpha^k = p\alpha^k;\ \ 0 < p < 1;\ \ k := k + 1$.

Theorem 2.2. The algorithm is finite.

Indeed, a point $x^*_{\sigma s}$ being found it is either optimal for the LCCM problem and after a number of iterations, taking into account (12), will satisfy the optimality criterion, or will be replaced by an optimal point, because α will satisfy the conditions of theorem 2.1.1.

2.3. The Cardinality Constraints Linear Programming problem (CCLP) is formulated as follows [10], [12]:

max cx

subject to:

$$Ax = b,\ x \geqslant 0$$

$$|x|^+ \leqslant r,\ r \leqslant \text{rank } A. \tag{30}$$

Using function σ, we can try to satisfy the cardinality constraint by formulating the σ CCLP problem as:

max cx, with conditions:

$$Ax = b,\ x \geqslant 0$$

$$\sum_{i=1}^{n} x_i/(\alpha_i + x_i) \leqslant r. \tag{31}$$

A general form of CCLP problem is the generalized lattice point problem (GLPP) [2], [5]:

max cx

subject to

$$Ax = b,\ x \geqslant 0$$

$$|\, x \,|_{\mathfrak{J}}^{+} \leqslant r,\ r \leqslant \text{rank } A,$$

where we denote by $|\, x \,|_{\mathfrak{J}}^{+}$ the number of strictly positive components, of x, those indices are in a specified index set $\mathfrak{J}$.

Let now be σGLPP problem defined as:

max cx

with conditions

$$Ax = b,\ x \geqslant 0$$

$$\sum_{i \in \mathfrak{J}} x_i/(\alpha_i + x_i) \leqslant r,\ r \leqslant \text{rank } A \tag{33}$$

We have

Theorem 2.3.:

There exists α_i such that a solution to σGLPP (σCCLP) problem is a solution to GLPP (CCLP) problem.

The theorem is an immediate consequence of properties (11), (12) of function σ i.e. to be decreasing with respect to α.

We want to point out in what follows, CABOT's results, as they are obtained by using a function quite similar to the function introduced by us in this paper.

Victor Cabot propounds in [2] to find out a feasible solution to GLPP problem. To this end, he considers the problem VC:

$$max\ y = \sum_{i \in \mathfrak{J}} 1/(\alpha + \beta x_i),\ Ax = b,\ x \geqslant 0,\ \alpha > 0,\ \beta > 0 \tag{34}$$

where

$$\alpha < \beta M\varepsilon/[rM - (r+1)\varepsilon],\ \varepsilon = 1/D \tag{35}$$

and where D is the largest absolute value of the determinants of the basis matrices for all basic solutions to $Ax = b$, $x \geqslant 0$, and M is an upper bound of x, for any feasible x so that $\alpha > 0$.

Cabot's function, which is a convex one, has the property of having greater value in the verteces of the polyhedron of feasible solutions to GLPP problem than in the verteces which are not feasible to GLPP problem. Two theorems are given:

Theorem VC1: The optimal solution to VC problem is a feasible solution to GLPP, if there is a feasible soluton to GLPP.

Theorem VC2: Any feasible solution to GLPP problem is a feasible solution to VC problem, with:

$$y \geqslant \{r/(\alpha + \beta M)\} + \{(|\mathcal{J}| - r)/\alpha\} \tag{36}$$

where the number of indices in $\mathcal{J}$ is denoted by $|\mathcal{J}|^+$.

The similarity of these results with theorem 2.3. is obvious.

2.4. Linear Programming Problem Minimizing the Total Cost Including the Set up or the Fixed Charge Cost for starting the fabrication (FCLP).

This problem, well known in the literature, is formulated as follows:

$$min\ z = \sum_{i=1}^{n} f_i(x_i),\ Ax = b,\ x \geqslant 0$$

where

$$f_i(x_i) = c_i x_i + M_i \delta_i,\ M_i > 0,\ \delta_i = \begin{cases} 0, \text{ if } x_i = 0 \\ 1, \text{ if } x_i > 0. \end{cases}$$

Using function $\delta_i(x_i)$, FCLP problem becomes σ FCLP:

$$min\ \sigma_{FCLP} = \sum_{i=1}^{n} c_i x_i + \sum_{i \in \mathcal{J}} M_i \delta_i(x_i),\ Ax = b,\ x \geqslant 0 \tag{37}$$

2.5. Problem with Constraints {zero, k} on the Variables.

Let be a mathematical programming problem with constraints on the variables of the form $x_i \in \{0, k\}$ for $i \in \mathcal{J}$. Such conditions may be written under the form:

$$\begin{aligned} x_i &\geqslant k x_i/(\alpha_i + x_i),\ i \in \mathcal{J} \\ 0 &\leqslant x_i \leqslant k, \qquad i \in \mathcal{J}. \end{aligned} \tag{38}$$

It is easily seen that the first constraints are satisfied, either for $x_i = 0$, or for $x_i \geqslant k - \alpha_i$. Hence, by correspondingly selecting α_i, we can justify conditions (38), as equated to the integrity conditions $x \in \{\text{zero}, k\}$. We think that such conditions could be used especially in connection with a SUMT technique.

Let us remark now that CCLP problem may be modelled

$$\begin{gathered} max\ cx,\ Ax = b,\ x \geqslant 0 \\ x_i - M_i z_i \leqslant 0,\ z_i \geqslant \frac{z_i}{\alpha_i + z_i} \\ 0 \leqslant z_i \leqslant 1 \\ i = 1, \ldots, n \\ \sum_{j=1}^{n} z_j \leqslant r \end{gathered} \tag{39}$$

where M_i is an upper bound of x_i.

2.6. Disjunctive Constraints-Problems.

A constraint of the form $x_i \in [l_i,\ l_i + \alpha_i]$ or $x_i \in [L_i - \alpha_i,\ L_i]$ is equivalent to conditions

$$\begin{cases} x_i^2 - (l_i + L_i)\, x_i + (l_i + \alpha_i)\,(L_i - \alpha_i) \geqslant 0 \\ l_i \leqslant x_i \leqslant L_i \\ \alpha_i > 0. \end{cases} \tag{40}$$

Constraints (40) are a generalization of constraints (38). For $\alpha_i > 0$ sufficiently small, constraints (40) are the continuous equivalent of discrete condition

$$x_i \in \{l_i,\ L_i\}.$$

Thus, a problem $min\ \{cx \mid Ax \leqslant b,\ x_i = l_i$ or $x_i = L_i\ i_i \in \mathfrak{J}\}$ is equivalent for M large enough to:

$$min\ \{cx - \sum_{i \in \mathfrak{J}} M_i[x_i^2 - (l_i + L_i)\, x_i + (l_i + \alpha_i)\ (L_i - \alpha_i)] \mid Ax \leqslant b,$$

$$l_i \leqslant x_i \leqslant L_i,\ \alpha_i > 0\}.$$

For $l_i = 0$, $L_i = 1$, $\alpha \to 0$, we refind Raghavachari's result [9].

2.7. The Set Covering Problem (SCP), intensively studied in the last time, due to its numerous applications, is enunciated as follows:

$$min\ \{cx \mid Ax \geqslant e,\ x_j = 0 \text{ or } x_j = 1,\ j \in N\}$$

where $A = (a_{ij})$ is $m \times n$, $e \in R^m$, $e = (1, \ldots, 1)$,

c and $x \in R^n$, $a_{ij} \in \{0,\ 1\}$, $i \in M = \{1, \ldots, m\}$, $j \in N = \{1, \ldots, n\}$.

The equivalent in real variables of the covering set problem (RSCP) can be formulated:

$min\ \{cx \mid Ax \geqslant e\}$ where A, x, e, c have the above specifications, and in addition:

$$\begin{gathered} x_j^2 - x_j + \alpha_j(1 - \alpha_j) \geqslant 0 \\ 0 \leqslant x_j \leqslant 1 \end{gathered} \tag{41}$$

or under a penalty form:

$$min\Big\{cx - \sum_{j=1}^{n} M_j(x_j^2 - x_j + \alpha_j(1 - \alpha_j)) \mid Ax \geqslant e,$$

$$0 \leqslant x_j \leqslant 1,\ j = 1, \ldots, n\Big\} \tag{42}$$

We can also formulate a real minimal covering problem (RMSC):

$$\begin{gathered} min \sum_{j=1}^{n} [x_j/(\alpha_j + x_j)] \\ Ax \geqslant e \\ x_j^2 - x_j + \alpha_j(1 - \alpha_j) \geqslant 0 \\ 0 \leqslant x_j \leqslant 1 \end{gathered} \tag{43}$$

analogously use formulate the real minimal set convering problem with equality constraints, RMSCE or the minimal set partitioning problem (RMSP) as:

$$min \sum_{j=1}^{n} [x_j/(\alpha_j + x_j)] \tag{44}$$

$$Ax = e,\ 0 \leqslant x_i \leqslant 1,\ x_i^2 - x_i + \alpha_i(1 - \alpha_i) \geqslant 0,\ i = 1, \ldots n$$

The set packaging problem (SP) defined as being:

$$max\ \{cx \mid Ax \leqslant e,\ x_i = 0 \text{ or } x_i = 1,\ \forall\ i \in N\}$$

has as an equivalent in real (RSP):

$$max \left\{ cx + \sum_{i=1}^{n} M_i(x_i^2 - x_i + \alpha_i(1 - \alpha_i)) \mid Ax \leqslant e, \right.$$

$$\left. 0 \leqslant x_i \leqslant 1,\ i = 1, \ldots, n \right\}. \tag{45}$$

The theoretical interest of the above formulation is without doubt, the computational one has to be seen.

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MODELES DE PROGRAMMATION A VARIABLES REELLES POUR LES PROBLEMES DISCRETS OU MIXTES

On propose pour certains problèmes à variables discrètes ou mixtes des modèles alternatifs en variables réelles, ayant pour base la fonction $\delta(x) = x/(\alpha + x)$, $\alpha > 0$, ou ses compositions: $\sigma(x) = \sum_i \delta_i(x_i)$, $\Pi(x) = \prod_i \delta_i(x_i)$, etc.

On montre, par exemple, qu'il existe un vecteur $\alpha > 0$, de telle façon que la solution du problème $min\ \sigma(x, \alpha) = \sum_{i \in I} x_i|\ (\alpha_i + x_i)$, $Ax = b$, $x > 0$ est une solution du problème $min\ |x|^+$, $Ax = b$, $x \geqslant 0$, où par $|\ x\ |^+$ on note le cardinal de x, soit le nombre de composantes strictement positives de x, offrant ainsi une autre alternative, à résultats plus forts, pour la procédure de Cabot (1975) de détermination en réel d'une solution admissible pour un problème de programmation linéaire généralisée sur une latice.

On propose aussi des modèles continus pour des problèmes à restrictions disjonctives, généralisant les résultats de Raghavachari (1969), ainsi que des modèles pour les problèmes de recouvrement minimal, pour celui des ensembles de partition minimale ou des ensembles d'empaquetage.

МОДЕЛИ ПРОГРАММИРОВАНИЯ С РЕАЛЬНЫМИ ПЕРЕМЕНЫМИ ДЛЯ ДИСКРЕТНЫХ И СМЕШАННЫХ ЗАДАЧ

В работе предлогаются альтернативные модели с реальными перемеными для некоторых задач с дискретными или смешанными перемеными, основанными на функции $\delta(x) = x/(\alpha + x)$, $\alpha > 0$ или на её сочетаниях

$$\sigma(x) = \sum_i \delta_i(x_i),\ \Pi(x) = \prod_i \delta_i(x_i)$$

Показывается, например, что существует вектор $\alpha > 0$, так, что решение задачи $min\ \sigma(x, \alpha) = \sum_{i \in I} x_i/(\alpha_i + x_i)$, $Ax = b$, $x \geqslant 0$ является решение задачи $min\ |\ x\ |^+$, $Ax = b$, $x \geqslant 0$, где $|x|^+$ — кардинал x, т.е. число строго позитивных компонентов x, давая таки образом новый вариант для процедуры Çabat (1975) определения в реале некоторого допустимого решения, обобщаемого на латиче.

Предлогаются также непрерывные модели для задач с разделительными ограничениями, обобщая результаты, полученные Raghavachari (1967), и модели для задачи минимального покрытия, для задачи минимальной партиции множества или упаковочного множества.

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