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Algorithm for the Decomposition of an Angular Shaped Matrix

Many economic problems can be solved through linear or linear-dynamic programming. Dantzig and Wolfe, Rosen et alii supposed a special form of the system, called angular shape. Nevertheless, so as to be able to apply those algorithms in economic problems, we must have the possibility of "cutting" the constraint matrix in the desired shape. This is what we intend to evince in this paper.

1. INTRODUCTION

Many economic problems can be modelled through a mathematical programme problem. But, either the linear-dynamic programming or merely the linear programming is involved, the problem as a whole, has very great dimensions, a fact that makes it very difficult for the researcher to work it out. Many algorithms were, of course, constructed for the linear-dynamic programming, as is, for instance, Glassey's operative production programming algorithm [4]; it proved to be less efficient, either from the computer-time viewpoint, or from the point of view of the sub-optimality, than Rosen's algorithm [2]. Rosen's algorithm, though initially conceived only for linear problems, proved to be also suitable, through the decomposition it performs, for linear-dynamic problems. But there are cases when the mathematical model is not structured in such a way as to ensure the applicability of Rosen's algorithm. To this end, we dealt with the decomposition of the total-constraint matrix, in such a way as to be able to apply afterwards Rosen's algorithm.

The fact that we determine only line-index sets, and not also column-index sets, may appear strange to the reader not familiar with computer-programming notions. But the total-constraint matrix is very seldom met with, and this fact means its storage by non-null elements, in the sense: line, column, value. Accordingly, by determining a certain line index as being block-index, for instance, we can easly find out, through programming, what column indices

are cooresponding to it. I leave open the problem of programming this algorithm for the case when storage waste could be avoided.

On this occasion, I express my most deep thanks to professor Corneliu Zidăroiu, for the afforded substantial assistance and coordination, which helped me to solve this problem.

2. THE PROBLEM

The intensive development of economy leads sometimes, for its analysis, to very great linear-dynamic or linear problems. There are cases when the problems which model an economic system cannot be solved with the existing programme packages, as for instance the OPALINE, due to their dimensions.

There are, at present, a series of algorithms, based upon the decomposition of the problem into interconnected subproblems, specialized for very great dimensions. One of the most efficient is Rosen's algorithm which, though initially conceived only for great-dimensions linear programmes, can be easily applied also to linear-dynamic problems.

The Rosen's algorithm starts from a special-shape constraint matrix, called angular form:

$$U = \begin{Vmatrix} A_1 & A_2 & \dots & A_p \\ B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_p \end{Vmatrix}$$

where matrices $A_1, A_2, \dots, A_p$ form the coupling matrix, and matrices B_1, $B_2, \dots, B_p$ are called blocks, the rest of elements being null.

The problem to be solved before programming Rosen's algorithm is as follows: Having a matrix whatever A, rare, of dimension $m \times n$ and order $A = m$, it is required to bring it, if possible, to the angular form.

We tried to solve this problem through the algorithm presented in this paper.

3. PRESENTATION OF THE METHOD

We bring in a relation (called kinship relation), existing between the line indices of matrix A, defined thus: the line index i is in kinship relation to the line index j, or iRj, if there is at least one column index K, such that $A(i, k) \neq 0$ and $A(j, k) \neq 0$.

We remark that relation R is reflexive, symmetrical, but not also transitive.

We construct, corresponding to relation R, a working matrix, called kinship-relation matrix:

$$S(i, j) = \begin{cases} 1, & \text{if } i \ R. \ j \text{ and } i \neq j \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We state that indices i and j from definition (1) are line indices for matrix A, while, in matrix S, index j becomes column index. We remark that matrix S has the dimension $m \times m$.

We select, within cycle 1, a sequence of line indices from matrix A, viz. $L_1 = \{x_1^1, x_2^1, \ldots, x_{\lambda_1}^1\}$, where $x_1^1 = 1$, $x_2^1 = x_1^1 + 1$ and $x_{\lambda_1}^1 + 1 > m$, $x_{\lambda_1}^1 \leqslant m$ the sequence $\{x_3^1, \ldots, x_{\lambda_1}^1\}$ is the Fibonacci number sequene, determined by the choice of x_1^1 and x_2^1.

We choose a set:

$$P^1 \subseteq . L^1$$

with

$$\text{card } P^1 > 1.$$

We construct, for set P^1, the following sets:

$$C_{p_1} = \{e_1 \in \{1, 2, \ldots, m\} / S(p_1, e_1) = 1\}, \quad \forall\, p_1 \in P^1, \tag{2}$$

$$B_{p_1} = \mathbf{C}\, C_{p_1} - \{\alpha_{p1}\} \text{ (possibly)}. \tag{3}$$

If

$$\bigcup_{p_1 \in P^1} C_{p_1} \subset \mathbf{C}\, P^1 \tag{4}$$

then P^1 forms part of the set of the coupling indices.

If

$$\bigcup_{p_1 \in P^1} C_{p_1} \supseteq \mathbf{C}\, P^1 \tag{5}$$

then P^1 forms the coupling indices set.

If

$$\bigcup_{p_1 \in P^1} B_{p_1} \subset \mathbf{C}\, P^1 \tag{6}$$

then P^1 forms the line-indices set of a block (which, possibly can be completed in the subsequent cycles).

Let us assume that, as a result of the achievement of cycle 1, we found the index sets P_i^1, $1 \leqslant i \leqslant i_1$, which represent, either coupling-index sets, or block-index sets. Let us suppose that

$$L_1 - \bigcup_{i=1}^{i_1} P_i^1 \neq \varnothing. \tag{7}$$

Then we shall call indices:

$$k \in L_1 - \bigcup_{i=1}^{i_1} P_i^1 \tag{8}$$

free indices and we shall build up the free-index set L:

$$L = L_1 - \bigcup_{i=1}^{i_1} P_i^1 \tag{9}$$

which will be possibly completed in the next cycles.

Let us suppose that we arrived to cycle α (where $\alpha \leqslant \left[\frac{m}{4}\right] + 1$ as we shall see subsequently) and we determined, on the previous cycles, sets P_i^1, $1 \leqslant i \leqslant i_1$, ..., $P_i^{\alpha-1}$, $1 \leqslant i \leqslant i_{\alpha-1}$.

The number sequence selected in cycle $\alpha - 1$ is $L^{\alpha-1}$. We look in set $L^{\alpha-1}$ for the first number $x_{d_{\alpha-1}}^{\alpha+1}$ which satisfies relation:

$$x_{d_{\alpha-1}+1}^{\alpha-1} \neq x_{d_{\alpha-1}}^{\alpha-1} + 1. \tag{10}$$

We construct the number sequence $L_\alpha = \{x_1^\alpha, x_2^\alpha, ..., x_{\lambda_\alpha}^\alpha\}$ where $x_1^\alpha = x_{d_{\alpha+1}}^{\alpha-1}$, $x_2^\alpha = x_1^\alpha + 1$, $x_{\lambda_\alpha+1}^\alpha > m$, $x_{\lambda_\alpha}^\alpha \leqslant m$, and the sequence $\{x_3^\alpha, ..., x_{\lambda_\alpha}^\alpha\}$ is the Fibonacci number sequence determined by x_1^α and x_2^α.

We take a subset whatsoever:

$$P^\alpha \subseteq L_\alpha \tag{11}$$

with

$$\text{card } P^\alpha > 1.$$

Let us suppose that, in the previous cycles, we determined sets D_i, $1 \leqslant i \leqslant r$, as forming parts of the coupling-index set.

If

$$\bigcup_{p_\alpha \in P^\alpha} C_{p_\alpha} \cup \overset{r}{\underset{i=1}{\cup}} \left(\bigcup_{d_i \in D_i} C_{d_i} \right) \subset \mathbf{C}\, P^\alpha \cup \left(\overset{r}{\underset{i=1}{\cup}} \mathbf{C}\, D_i \right) \tag{12}$$

then $P^\alpha \cup \left(\overset{r}{\underset{i=1}{\cup}} D_i \right)$ forms part of the coupling-index set.

If

$$\bigcup_{p_\alpha \in P^\alpha} C_{p_\alpha} \cup \overset{r}{\underset{i=1}{\cup}} \left(\bigcup_{d_i \in D_i} C_{d_i} \right) \supseteq \mathbf{C}\, P^\alpha \cup \left(\overset{r}{\underset{i=1}{\cup}} \mathbf{C}\, D_i \right) \tag{13}$$

then $P^\alpha \cup \left(\overset{r}{\underset{i=1}{\cup}} D_i \right)$ forms the coupling-index set, determined in a previous cycle.

Let F be a block-index set, determined in a previous cycle.

If

$$\left(\bigcup_{p_\alpha \in P^\alpha} B_{p_\alpha} \right) \cup \left(\bigcup_{f \in F} B_f \right) \subset \mathbf{C}\, P^\alpha \cup \mathbf{C} F \tag{14}$$

then $P^\alpha \cup F$ forms the set of the indices of a block.

Relation (13) may be used, either to complete a block-index set, previously found out with a view to determining the maximal block, but at

the same time, we can consider set F as void, and we determine, through relation (13), a new block.

Let us assume that we determined, in cycle α, sets P_i^α, $1 \leqslant i \leqslant i_\alpha$. If

$$L_\alpha - \bigcup_{i=1}^{i_\alpha} P_i^\alpha \neq \emptyset \tag{15}$$

then the indices from set $L_\alpha - \bigcup_{i=1}^{i_\alpha} P_i^\alpha$, called free indices, will be inserted in the free index, L, as defined above.

4. WAY OF SELECTING THE NUMBER SEQUENCES WITHIN EACH CYCLE

As already remarked, when going over to a new cycle $\alpha + 1$, the number sequence $L_{\alpha+1}$ depends on sequence L_α from the previous cycle in the sense that one selects:

$$x_1^{\alpha+1} = x_{d_\alpha}^\alpha + 1 \tag{16}$$

where $x_{d_\alpha}^\alpha$ is the first number from L_α which satisfies the following inequality:

$$x_{d_{\alpha+1}}^\alpha \neq x_{d_\alpha}^\alpha + 1. \tag{17}$$

We select then:

$$x_2^{\alpha+1} = x_1^{\alpha+1} + 1 \tag{18}$$

and sequence $\{x_3^{\alpha+1}, \ldots, x_{\lambda\alpha+1}^{\alpha+1}\}$ is, in fact, the Fibonacci number sequence, determined by $x_1^{\alpha+1}$ and $x_2^{\alpha+1}$ and which satisfies:

$$x_{\lambda_{\alpha+1}+1}^{\alpha+1} > m, \quad x_{\lambda_{\alpha+1}}^{\alpha+1} \leqslant m. \tag{19}$$

This way of selecting number sequences is valid until one arrives to cycle $\left[\frac{m}{4}\right]$. This cycle would comprise rather few numbers (line indices), if we would select the number sequence in the same way as yet, so we shall select a number sequence built up from all those line indices which did not enter any of the previous number sequences.

It is possible that the decomposition carried out so as to determine the line indices, corresponding to the blocks and to the coupling matrix, be achieved in this cycle $\left[\frac{m}{4}\right]$, But if there remain free indices from the $\left[\frac{m}{4}\right]$ cycles, we shall carry out a supplementary cycle, where the number sequence is the set L of free indices, determined in the previous cycles.

To the end of this supplementary cycle we can accurately say whether matrix A allows, yes or not, the angular shape.

Theorem 1.

The condition

$$\bigcup_{p \in P} C_p \subset \complement P \tag{20}$$

is necessary and sufficient for P to form part of the coupling index set.

Proof.

Let D be the coupling-index set and $F_1, F_2, ..., F_r$ the block-index sets. If $i \in D$, there are at least two indices $k_1, k_2 \in \{1, 2, ..., r\}$ such that, for indices $j_1 \in F_{k_1}$, $j_2 \in F_{k_2}$ we have $S(i, j_1) = 1$ and $S(i, j_2) = 1$.

As $P \subset D$, any element $p \in P$ satisfies the condition that there is a subset

$$I \subset \{1, 2, ..., r\}, \quad I \neq \emptyset \tag{21}$$

such that there are at least two indices $k_1, k_2 \in I$ for which indices $j_1 \in F_{k_1}$, $j_2 \in F_{k_2}$ satisfy $S(p, j_1) = 1$ and $S(p, j_2) = 1$.

Sets C_p, $p \in P$ are defined as follows

$$C_p = \{i \in \{1, 2, ..., m\} / S(p, i) = 1\}. \tag{22}$$

Hence, for a $p \in P$, set C_p defines the line indices with which line p is in kinship relation.

As P is part of the coupling-index set, its complementary is as follows:

$$\complement P = F_1 \cup F_2 \cup ... \cup F_r \cup (D - P). \tag{23}$$

The union of sets C_p, $p \in P$ defines the line indices with which indices $p \in P$ are in kinship relation.

It results, from the above reasons, that the indices from P are in kinship relation with part of the indices from P' s complementary. In other words, we proved that, if $P \subset D$, relation (20) is satisfied.

Let us prove now that, if relation (20) is satisfied, then $P \subset D$.

Let A be a non-void set, such that

$$\left(\bigcup_{p \in P} C_p\right) \cup A = \complement P. \tag{24}$$

We may consider set A as being

$$A = \bigcup_{f \in F} C_f \tag{25}$$

where

$$F \subset D. \tag{26}$$

Hence

$$\left(\bigcup_{p \in P} C_p\right) \cup \left(\bigcup_{f \in F} C_f\right) = \complement P. \tag{27}$$

From the fact that $F \subset D$ and that relation (27) is satisfied, it results that P is part of the coupling indices, hence

$$P \subset D$$

O.e.d.

Theorem 2

Condition

$$\bigcup_{p \in P} C_p \supseteq \mathbf{C}\, P \tag{28}$$

is necessary and sufficient for P to build up the coupling index-set.

Proof

If P is the coupling index set, and $F_1, F_2, ..., F_r$ are the block-index sets, we showed in the proof of theorem 1 that, for a $p \in P$, there are at least two indices $k_1, k_2 \in \{1, 2, ..., r\}$, such that for indices $j_1 \in F_{k_1}$, $j_2 \in F_{k_2}$ we have $S(p, j_1) = 1$ and $S(p, j_2) = 1$.

As we assumed that P forms the whole set of the coupling indices, it results that the totality of indices $p \in P$ is in kinship relation with the totality of indices of the complementary; there may be also kinship relations between the indices from P. This means, taking into account the definition of sets C_p, $p \in P$, that relation (28) is satisfied.

Reciprocically let A be a set, possibly void, such that:

$$\bigcup_{p \in P} C_p = \mathbf{C}\, P \cup A. \tag{29}$$

We may consider:

$$A \subset P. \tag{30}$$

As we showed in the proof of theorem *1*, the union of sets C_p, $p \in P$ defines the line indices, with which $p \in P$ is in kinship relation.

It results, from relations (29) and (30), that indices $p \in P$ are in kinship relation with all indices in $\mathbf{C}\, P$ and with the indices in $A \subset P$.

As the line indices of a block, even if they are in kinship relation between themselves, cannot be in kinship relation with the line indices of the other blocks (the indices which are in the complementary), and as the coupling-index set is, compulsorily, in kinship relation with all block indices, hence with the indices in the complementary and there are, possibly, kinship relations between the coupling indices, it results the set P forms the coupling. index set.

Q.E.D.

Theorem 3

Condition

$$\bigcup_{p \in P} B_p \subset \mathbf{C}\, P \tag{31}$$

is necessary and sufficient for P to form the set of the indices of a block.

Proof

Let us assume that D forms the coupling-index set and $F_1 F_2, ..., F_r$, forms the sets of the block indices. Let $e \in \{1, 2, ..., r\}$ be, such that $P = F_e$.

For any $k_1 \in \{1, 2, ..., r\} - \{i\}$ and any $j_1 \in F_1$, $j_2 \in F_e$, we have $S(i_1, j_2) = 0$. At the same time, there is at least one index $i \in D$ such that $S(i, j_2) = 1$.

Sets B_p, $p \in P$ are defined as follows:

$$B_p = \mathbf{C}C_p - \{p\} \tag{32}$$

$$B_p = \{e \in \{1, 2, ..., m\} / S(p, e) = 0, \; p \neq e\}. \tag{33}$$

Sets B_p define the line indices with which $p \in P$ is not in kinship relation.

A P forms a block, the union of sets B_p, $p \in P$ is the line-index set of the other blocks.

As we shoved above, there is at least one index $i \in D$ which is in kinship relation with indices $p \in P$.

All those prove that the union of sets B_p, $p \in P$ must be included within the complementary of P, because that union is disjoint from set D.

Let us suppose, reciprocally, that relation (31) is true. Then, there is a set A, non-void, such that:

$$A \subset \mathbf{C}P, \quad A \subset D \tag{34}$$

and

$$\bigcup_{p \in P} B_p = \mathbf{C}P - A. \tag{35}$$

It results from relation (35) that indices $p \in P$ are not in kinship relation with indices $\mathbf{C}P - A$, but they are in kinship relations with the indices from A. If P would have been the coupling-index set, then indices $p \in P$ would have been in kinship relation with all indices in P's complementary and this contradicts the fact that the indices in P' are not in kinship relation with the indices in $\mathbf{C}P - A$. Hence, considering set A as part of the coupling indices, it results that P forms the set of the indices of a block.

Q.E.D.

Theorem 4

If after carrying out the supplementary cycle, there remain free indices, then the original matrix does not allow decomposition in angular form.

Proof

The selection of indices in each cycle, the tests they are submitted to, be they both indices in a cycle, or indices left free from previous cycles, determines in a unique manner a batch of sets. If there remain, after testing the necessary and sufficient condition for determining those sets over all indices (at the end of cycle $\left[\frac{m}{4}\right]$ there remain nevertheless free indices, those indices

will build up number sequences of the supplementary cycle. We resume the tests and if, after these tests there still remain some free indices, this means that certain lines cannot be, within the matrix, neither coupling lines, nor block lines, hence the decomposition cannot take place.
Q.e.d.

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ALGORITHME POUR DECOMPOSER UNE MATRICE DE FORME ANGULAIRE

Bon nombre de problèmes peuvent être résolus à l'aide de la programmation linéaire ou linéaire-dynamique. Dantzig et Wolfe, Rosen o.a. ont supposé le système comme ayant une forme particulière, nommée forme angulaire. Mais pour pouvoir appliquer ces algorithmes aux problèmes économiques, il faut avoir la possibilité de "couper" la matrice restriction selon la forme souhaitée. C'est le but de cet article.

АЛГОРИТМ РАЗЛОЖЕНИЯ МАТРИЦ С УГОЛНОЙ ФОРМОЙ

Многие экономические задачи могут быть решены при помощи линейного или линейно-динамического программирования. Данциг и Волф, Розен и. д. отнесились к специальной форме системы, а именно к уголной форме. На, для того, чтобы могли применять, эти алгоритмы в решение экономических задач необходима возможность «разреза» матрицы ограничений в нужную форму. И это является тем, что хотим показать в настоящей статьи.

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