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Synthesis and Decision by Deffuzzification

This paper presents a new approach to the fuzzy programming problem by taking into account the recent results in the field of multiple descriptions synthesis.

1. INTRODUCTION

Let $\mathcal{F}(x)$ be the set of all fuzzy sets in a non empty set X, i.e. the set of all functions $f : X \to [0, 1]$.

It is well known (see Zadeh [11]) that for any $f, g \in \mathcal{F}(x)$ the operations $\wedge$ and $\vee$ defined by

$$(f \wedge g)(x) = \min_{x \in X} (f(x),\ g(x)),$$

$$(f \vee g)(x) = \max_{x \in X} (f(x),\ g(x))$$

induce a lattice structure into $\mathcal{F}(x)$.

Let S be a set of unrelated statements. The set $\mathcal{F}(S)$ can be regarded as the set of functions μ assigning the truth value to any statement of S. If $s, t \in S$, then the truth values of the compound statements "s and t" and "s or t" are expressed by

$$\mu(s \text{ and } t) = \min (\mu(s),\ \mu(t)),$$

$$\mu(s \text{ or } t) = \max (\mu(s),\ \mu(t))$$

(see Bellman and Giertz [1]). We recall that the value $f(x)$, for $x \in X$, is ths degree of membership of x and can indicate our willingness to accept x as a member in $\mathcal{F}(X)$.

Let now Δ a real function defined in the interval [0, 1], strictly increasing in $\left[0, \frac{1}{2}\right]$ and such that $\Delta(0) = \Delta(1) = 0$ and $\Delta(a) = \Delta(1 - a)$, $a \in [0, 1]$. (Function Δ was introduced by Knopfmacher [6] in order to state a general form of the entropy of a measurable fuzzy set.) Di Nola and Ventre [4] define a new lattice $L = (\mathcal{F}(X), \cap, \cup)$ in the set $\mathcal{F}(X)$ by means of the operations

$$(f_1 \cap f_2)(x) = f_1(x) \cap f_2(x) = f_i(x)$$

such that either $\Delta(f_i(x)) < \Delta(f_j(x))$ or $\Delta(f_1(x)) = \Delta(f_2(x))$ implies $f_i(x) \leqslant f_j(x)$, $i \neq j$, $i, j = 1, 2$;

$$(f_1 \cup f_2)(x) = f_1(x) \cup f_2(x) = f_i(x)$$

such that either $\Delta(f_i(x)) > \Delta(f_j(x))$ or $\Delta(f_1(x)) = \Delta(f_2(x))$ implies $f_i(x) \geqslant f_j(x)$, $i \neq j$, $i, j = 1, 2$.
When $X = S$, if for any s, $t \in S$ we set

$$\mu(s \text{ and } t) = \mu(s) \cap \mu(t),$$

$$\mu(s \text{ or } t) = \mu(s) \cup \mu(t)$$

in order to assign the truth value of the statements "s and t" and "s or t", then the truth value of the one of two statements which carries a greater (less) amount of information — according to De Luca and Termini [2] and Knopfmacher [6] — can give the willingness to accept "s and t" ("s or t"). Of course, such a model holds when an entropy is defined in $\mathcal{F}(S)$, for instance under Knopfmacher hypotheses [6].

In the following sections we shall deal with the role of the operation $\cap$ in some processes which reduce entropy of a given fuzzy set, i.e. some "defuzzification" processes. For instance, a sequence $f_{A_1}, \ldots, f_{A_n}$ of fuzzy sets such that f_{A_j} is a "sharpened version" of f_{A_i}, $i < j$, represents a defuzzification process. In this case we are not sure that the willingness to accept x as a member of A_j is not less than the willingness to accept x as a member of A_i: the borderline of the set A_j is clearer than that one of A_i. Our aim is just to study some processes which contribute to clarify if an element is in a given set or not.

2. SYNTHESIS OF A MULTIPLE DESCRIPTION

Before applying the above considerations to a synthesis process of more than one description of a given object, we recall some notions. (More detailed problems and a deeper treatment of synthesis concept can be found in Negoita [7], [8], [9]).

Let x be an element of X. According to De Luca and Termini [3], a function $\varphi : x \to (f_1(x), \ldots, f_n(x))$, $n > 1$, where $f_i \in \mathcal{F}(X)$, $i = 1, \ldots, n$, is said to be a *multiple description* of the element x. The description of an object by means of more than a single property finds its set theoretical counterpart in the function φ. Negoita [7] defined a *synthesis process* of the n-tuple $(f_1(x), \ldots, f_n(x))$ as a new description given by $f(x) = \wedge f_i(x)$. All desciptions

of x concour to define the new fuzzy set f. The acquainted knowledge is consistent with the former ones because the truth value of the statement "$x \in X$" is not greater than $f_i(x)$, for any i. This is a way to solve the conflict arising from several inexact descriptions of the same object. We just observe that this synthesis process does not give rise necessarily to a fuzzy set with entropy not greater than those ones of $f_1, .., f_n$.

Consider now the n-tuple $(f_1, ..., f_n)$ of fuzzy sets in X and the function $s : \mathcal{F}^n(X) \to \mathcal{F}(X)$ defined by

$$s(f_1(x), ..., f_n(x)) = \bigcap_{i=1}^{n} f_i(x) = f_j(x),$$

with $j \in \{1, .., n\}$ such that $\Delta(f_j(x)) \leqslant \Delta(f_h(x))$, for any $h \neq j$, and if $\Delta(f_j(x)) = \Delta(f_k(x))$ for some $k \neq j$, then $f_j \leqslant f_k$. We call *d-synthesis* the function s. By means of s an integration of the criteria f_i's is made, fuzzy set s is a new criterion which creates new knowledge. This definition of synthesis can be useful to solve a defuzzification problem arising from a multiple description, in fact the entropy (if it can be defined) of the fuzzy set s is not greater than those of all f_i s.

We stress that the two concepts of synthesis coincide if $\Sigma_i f_i(x) = 1$, i.e. the fuzzy sets $f_1, ..., f_n$ represent orthogonal properties defined on X. Indeed, if $f_j(x) = \min_i \{f_i(x)\}$, them $f_j(x) \leqslant \frac{1}{2}$. Furthermore if there would exist $f_h(x)$ such that $f_j(x) < f_h(x)$ and $\Delta(f_h(x)) < \Delta(f_j(x))$, then $f_h(x) > \frac{1}{2}$ and $f_h(x) > 1 - f_j(x)$. Therefore $f_j(x) + f_h(x) > 1$, which contradicts the hypothesis. (Further properties of d-synthesis are stated by Di Nola and Ventre [5] also in connection with a stability problem in fuzzy systems theory).

3. DECISION MAKING IN THE LATTICE L

We recall that a *fuzzy decision* is a fuzzy set $f_D = f_C \wedge f_G$, where f_C and f_G are fuzzy sets called *fuzzy constraint* and *fuzzy goal*, respectively. The problem of determining the maximizing decision

$$\sup_{x \in X} f_D(x) = \sup_{x \in X} ((f_C \wedge f_G)(x)) \tag{1}$$

is the *Fuzzy Mathematical Programming* (FMP) problem. The topic is developed, for intance, in the book of Negoita and Ralescu [10].

According to the meaning of the operations of the lattice L, the decision $f_D(x)$ in this lattice is defined as $f_D(x) = (f_C \cap f_G)(x)$ and the *optimizing decision* in L is

$$\bigcap_{x \in X} (f_D(x)) = \bigcap_{x \in X} ((f_C \cap f_G)(x)). \tag{2}$$

In other words an optimizing decision in L is a value of $(f_C \cap f_G)(x)$ such that makes minimum function Δ and if such two values exist, then the opti-

mizing decision is the minimum. In symbols, if $Y = f_C(X) \cup f_G(X)$, the optimizing decision in L is given by $min\ \{\bar{y} \in Y : \Delta(\bar{y}) \leqslant \Delta(y)$, for any $y \in Y)\}$. The optimizing problem (2) is solved, for instance, by finding the alternatives $\bar{x} \in X$ in which the values of entropy are minimal.

We shall find the value of the optimizing decision defined by (2).

Let put $A = \left\{a \in \left[0, \frac{1}{2}\right] : \left[\frac{1}{2} - a,\ \frac{1}{2} + a\right] \supseteq Y\right\}$ and

$$\bar{a} = \bigwedge_{a \in A} a,$$

$$b^- = \frac{1}{2} - \bar{a}, \quad b^+ = \frac{1}{2} + \bar{a}.$$

It easy to see that at least one of two values b^- and b^+ is an extremum of the set Y. Indeed, setting

$$e = inf\ Y, \quad e' = sup\ Y,$$

$$r = \left| e' - \frac{1}{2} \right|, \tag{3}$$

$$s = \left| \frac{1}{2} - e \right|, \tag{4}$$

if $r > s$, then $s \notin A$ and $r \in A$, and

$$e' \geqslant \frac{1}{2}. \tag{5}$$

For $r' < r$ it is $r' \notin A$ and for any $r'' > r$ it is $r'' \in A$. Thus $r = \bar{a}$ and, from (5), $r = e' - \frac{1}{2}$ and $e' = b^+$.

If $r < s$, then analogously $e = b^-$. Finally, if $r = s$, then either $e = b^-$ and $e' = b^+$, or $e = e' = b^+$, or $e = e' = b^-$.

We are now able to prove that the value of the optimizing decision expressed by (2) is either b^+ or b^-. In fact, for any $x \in X$ it is $f_C(x), f_G(x) \in \in [b^-, b^+]$ and as

$$\Delta(f_C(x)) \geqslant \Delta(b^-) = \Delta(b^+)$$

and

$$\Delta(f_G(x)) \geqslant \Delta(b^-) = \Delta(b^+)$$

we get

$$\Delta(f_C(x) \cap f_G(x)) \geqslant \Delta(b^-) = \Delta(b^+),$$

for any $x \in X$. Then, by (2),

$$\Delta\left(\bigcap_{x \in X} f_D(x)\right) \geqslant \Delta(b^-) = \Delta(b^+). \tag{6}$$

On the other hand, by definition of optmizing decission in L, we have

$$\Delta(\bigcap_{x\in X} f_D(x)) \leqslant \Delta(y),$$

for any $y \in Y$, and in particular

$$\Delta(\bigcap_{X\in x} f_D(x)) \leqslant \Delta(b^-) = \Delta(b^+).$$

Then, by (6), $\Delta(b^-) = \Delta(b^+) = \Delta(\bigcap_{x\in X} f_D(x))$ and, finally, $\bigcap_{x\in X} f_D(x) \in \{b^-, b^+\}$. Thus if

$$b^- \in Y \quad \text{and} \quad b^+ \in Y,$$

or

$$b^- \in Y \quad \text{and} \quad b^+ \notin Y,$$

then $\bigcap_{x\in X} f_D(x) = b^-$ and if

$$b^- \notin Y \quad \text{and} \quad b^+ \in Y,$$

then $\bigcap_{x\in X} f_D(x) = b^+$.

The above result is equivalent to the following equality

$$\bigcap_{x\in X} (f_D(x)) = (b^- \vee (1 - C_Y(b^-)) \wedge (b^+ \vee (1 - C_Y(b^+)),$$

where C_Y is the characteristic function of the set Y.

4. FIX POINT PROPERTIES

A maximizing decision in the lattice $\mathcal{L} = (\mathcal{F}(X), \wedge, \vee)$ is determined by a maxmin process (see Eq. (1)). Of course the maximum decision set of the FMP problem (1) does not coincide, in general, with the set of optimizing decisions in the lattice L. This appears evident from the definitions of latticeal operations $\wedge$ and $\cap$, i.e. from the meanings of the compound statements $P(x) \wedge (Qx)$ and $P(x) \cap Q(x)$ hinted in the sec. 1. For instance, if f_1 and f_2 are fuzzy sets in X and if f_1 is decreasing and f_2 is increasing in X, and if there exists $x_0 \in X$ such that $f_1(x_0) = f_2(x_0)$, then the FMP problem (1) with $f_C = f_1$ and $f_G = f_2$ has x_0 in the maximum decision set as easily one can verify.

We now shall show that it is

$$\Delta(f_i(x_0)) \geqslant \Delta((f_1 \cap f_2)\,(x_0)), \tag{7}$$

for any $x \in X$, $i = 1, 2$, i. e., x_0 belongs to the set of the "worst" decisions in the lattice L. We shall distinguish two cases in order to prove inequality (7).

i) $f_i(x_0) \leqslant \dfrac{1}{2}, \quad i = 1, 2.$

If $x' < x_0$, then $f_2(x') \leqslant f_2(x_0) \leqslant \frac{1}{2}$ and therefore

$$\Delta(f_2(x')) \leqslant \Delta(f_1(x_0)) = \Delta(f_2(x_0)) \text{ and } \Delta((f_1 \cap f_2((x')) \leqslant \Delta(f_2(x')) \leqslant \Delta(f_i(x_0)).$$

If $x'' > x_0$, then $\frac{1}{2} \geqslant f_1(x_0) \geqslant f_1(x'')$, and therefore $\Delta(f_i(x_0)) \geqslant$

$$\geqslant \Delta(f_1(x'')) \geqslant \Delta((f_1 \cap f_2)(x'')).$$

ii) $f_i(x_0) > \frac{1}{2}, \quad i = 1, 2.$

The proof is analogous.

Observe that under previous hypotheses if f_1 and f_2 are strictly monotone, then inequality (7) holds too, but the equality in (7) holds if and only if $x = x_0$.

Consider now that if f is a decreasing fuzzy set in $[0, 1]$ having a fix point x_0, then inequality (7) applies, when $f = f_1$ and $f_2(x) = x$, and it follows

$$\Delta(f(x_0)) = \Delta(x_0) \geqslant \Delta(f(x) \cap x), \quad \forall x \in X, \tag{8}$$

i.e., function $\Delta \circ f$ takes its maximum value in the fix point of f.

The inverse result holds under stronger hypotheses.

Let f be a continuous strictly decreasing fuzzy set in $[0, 1]$ such that

$$\Delta(f(x_0) \cap x_0) \geqslant \Delta(f(x) \cap x) \tag{9}$$

for any $x \in [0, 1]$ and for a suitable x_0. Then x_0 is the fix point of f.

Indeed it is well known that f has exactly one fix point $\bar{x}$. Then from the above result

$$\Delta(f(\bar{x}) \geqslant \Delta(f(x) \cap x),$$

for any $x \in [0, 1]$ and inequality (9) implies

$$\Delta(x_0 \cap f(x_0)) = \Delta(\bar{x}) \cap f(\bar{x})) = \Delta(\bar{x}). \tag{10}$$

Two cases can be considered.

i) $x_0 \cap f(x_0) = x_0$.

Then, from (10), $\Delta(x_0) = \Delta(\bar{x})$. Suppose $x_0 \neq \bar{x}$. If $\bar{x} < \frac{1}{2}$, then $x_0 > \frac{1}{2}$ and $f(x_0) < f(\bar{x}) = \bar{x} < \frac{1}{2}$.

Thus $\Delta(f(x_0)) < \Delta(\bar{x}) = \Delta(x_0')$ which contradicts the assumption in this case. So $x_0 = \bar{x}$. If $\bar{x} > \frac{1}{2}$ one can proceed analogously in order to get $x_0 = \bar{x}$ too. If $\bar{x} = \frac{1}{2}$, then equality $\bar{x} = x_0$ is immediate.

ii) $x_0 \cap f(x_0) = f(x_0)$.

Then, by (10),

$$\Delta(f(x_0)) = \Delta(\bar{x}). \tag{11}$$

Suppose $f(x_0) \neq \bar{x}$. If $\bar{x} < \frac{1}{2}$, then $f(x_0) > \frac{1}{2}$, $f(x_0) > \bar{x} = f(\bar{x})$ and, as f is strictly decreasing, $x_0 < \bar{x} < \frac{1}{2}$. From this $\Delta(x_0) < \Delta(\bar{x})$ and, by (11), $\Delta(x_0) < \Delta(f(x_0))$, which contradicts the assumption in this case. Thus $\bar{x} = f(x_0)$ and $\bar{x} = x_0$. If $\bar{x} \geqslant \frac{1}{2}$, one finds $\bar{x} = x_0$ too.

5. CONCLUDING REMARKS

Several problems in a fuzzy environment can be solved in the lattice L even if the construction in L of a doubling of the theory existing in the lattice $\mathcal{L} = (\mathcal{F}(X), \wedge, \vee)$ is impossible. The different structures of $\mathcal{L}$ and L and the fuzzy sets theoretical meaning of the operations determine non analogous applications. Let's give some examples. It seems quite natural to assume, as it happens in $\mathcal{L}$, $\overline{f(x)} = 1 - f(x)$ as a definition of complement in L. Then L is not a soft algebra; instead of De Morgan laws the following ones hold

$$\overline{f(x) \cap g(x)} = \overline{f(x)} \cap \overline{g(x)},$$

$$\overline{f(x) \cup g(x)} = \overline{f(x)} \cup \overline{g(x)}.$$

Furthermore we observe that an entropy d induces in a suitable subset of $\mathcal{F}(X)$ an ordering $\lesssim$ such that $f \lesssim g$ if and only if $d(f) \leqslant d(g)$. This ordering, which induces a total ordering by classes, is less fine that one "is a sharpened version of", defined in $\mathcal{F}(X)$ (see De Luca and Termini [2]). The ordering in L determined by operations $\cap$ and $\cup$ has some intermediate characters between the above ones: it is implied by the relation "is a sharpened version of" and implies the total ordering by classes. This ordering in L has the advantage of constructing some chains in L, which can be considered as models of coherent cognitive processes (see Di Nola and Ventre [4]) or as stable fuzzy systems (see Di Nola and Ventre [5]).

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SYNTHESE ET DECISION PAR DEFUZZYFICATION

L'article présente une nouvelle approche du problème de la programmation fuzzy tenant compte des résultats récents dans le domaine de la synthèse des descriptions multiples.

СИНТЕЗ И РЕШЕНИЯ С ПОМОЩЬЮ РАЗФУЗЗИФИЦИРОВАНИЯ

В работе представляется новый подход к вопросу программирования «фуззи» на основе некоторых недавних результатов, полученных в области синтеза множественных описаний.

8. 01. 1980