D. M. ENĂCHESCU

Computation Centre
University of Bucharest

Digital Methods in Estimating Spectral Densities

This work surveys the problems connected to the estimation of spectral density of time series. The important results in this field have been synthetized in three methods, transformation method, filtering method and auto-regressive estimation method.

I. INTRODUCTION

The probabilistic theory of time series enjoys today an extensive development, based upon the pioneer works of Wiener, Hincin, Kolmogorov, Cramer, Loève and Karhunen from the thirties and forties. At the same time as the development of this theory, arised the problem of drawing up a data manipulation procedure, with a view to synthetizing stochastic models, which could be used in describing and sometimes in controlling the mechanisms which generate time series.

The present paper, following the same line, is in fact a survey of the important results of the time series spectre estimation theory, with a view to setting up a programme library for the spectral analysis of the empirical time series — a library momentarily lacking from the applicative software of the FELIX C-256 computer.

We synthetised the important results from the field we refer to, under the form of three approaching methods of estimating the empirical dynamic series, viz.:

a) the indirect or transformation method, in which a Fourier transform weights the covariance, in order to estimated the spectre;

b) the direct or filtering method, which estimate the spectre as dispersion and the covariance of some filtered dynamic series;

c) the self-regressive estimation method.

a) Theoretical considerations

Let $x_1, x_2, ..., x_N$ be a selection on a real time series with $Mx_n = 0$, $Mx_n^4 < \infty$. We know from the time series statistical theory that, when the spectre has jumps, it cannot be generally consistent estimated. As the singular part of the spectre is of little practical importance, we shall consider in what follows *only absolutely continuous sectres with* $f(\lambda)$ the *spectral density.*

The transformation method for estimating the spectral density of a time series is based upon the following theorem:

Theorem 1. [1]. Let x_n be a weak stationary process time(series). If λ_1 and λ_2 are continuity points of the spectral function F, then

$$\lim_{n\to\infty} \int_{\lambda_1}^{\lambda_2} \frac{1}{n+1} \left| \sum_{j=0}^{n} x_j \, e^{-2\pi i j y} \right|^2 dy = F(\lambda_2) - F(\lambda_1). \tag{2.1}$$

Important remark. It is not correct to deduce from (2.1) that

$$\Phi_n(\lambda) = \frac{1}{n+1} \left| \sum_{j=0}^{n} x_j e^{-2\pi i j} \right|^2$$

is an estimation of the spectral density $f(\lambda)$. Indeed, let x_n be a real Gaussian process with $Mx_n = 0$ and $Mx_n^2 = 1$. In this case

$$R(0) = 1, \quad R(n) = 0 \quad n \neq 0 \quad \text{and} \quad f(\lambda) = 1.$$

Estimator $\Phi_n(0)$ becomes the square of a Gaussian random variable with zero mean and dispersion one, which cannot converge to $f(0) = 1$ when $n \to \infty$. The same argument is referred to also for other values of λ.

On the basis of relation (2.1) and of the above remark we are lead to take into account estimators of the form

$$f_N^*(\lambda) = \frac{1}{N} \sum_{k,j=1}^{N} x_k \bar{x}_j \int_{-1/2}^{1/2} e^{2\pi i (k-j)} w_N(y) \, dy \tag{2.2}$$

of the spectral density $f(\lambda)$. We shall call such *estimators* as *spectrographs.*

Note. We limit ourselves to this type of estimator, because as shown in [2], we can associate, to each type of estimator α_N^* of the spectral density, a spectrograph estimator with the same displacement as α_N^*, and which converges, in quadratic mean, between $f(\lambda)$ not worse than α_N^*.

The transformation method (called so, because the estimators are weightings of Fourier transforms of the convariance) studies the properties of the estimators of the form (2.2).

Theorem 2 [2]. The estimators of the form (2.2) are consistent estimators of the spectral density $f(\lambda)$.

Theorem 3 [2]. The displacement of the estimator (2.2) is

$$b_N(\lambda) = 0\left(\sqrt{\int_0^{1/2} (X - |\lambda|)^2 w_N(x) \, dx} \right) + 0\left(\frac{\log N}{N} \right) \tag{2.3}$$

Estimation of the spectral density through the transformation method

Weighting function $w_N(x)$ "spectral window"	Consistent estimator $f_N^*(\lambda) = \sum (C_j/N)\, w_j^{(N)}$
1. Bartlett $\frac{1}{2m}\left\{\frac{\sin^2 m\pi(\lambda - x)}{\sin^2 \pi(\lambda - x)} + \frac{\sin^2 m\pi(\lambda + x)}{\sin^2 \pi(\lambda + x)}\right\}$	$\sum_{j=-m+1}^{m+1}\left(1 - \frac{\lvert j\rvert}{m}\right)\frac{C_j}{N}\cos 2\pi j\lambda$
2. Rectangular estimation $\begin{cases} 1/4\,h, & \lvert x \pm \lambda\rvert < h \\ 0, & \text{remained} \end{cases}$	$\sum_{j=-N}^{N}\frac{C_j}{N}\frac{\sin 2\pi j h}{jh}\cos 2\pi jh$
3. Truncation of the periodogram (has a small displacement and dispersion) $\frac{1}{2}\left\{\frac{\sin 2\pi\left(m+\frac{1}{2}\right)(x-\lambda)}{\sin\pi(x-\lambda)} + \frac{\sin 2\pi\left(m+\frac{1}{2}\right)(x+\lambda)}{\sin\pi(x+\lambda)}\right\}$	$\sum_{j=-m}^{m}\frac{C_j}{N}\cos 2\pi jh$
4. Tukey $\frac{1}{2}\left\{0.54\frac{\sin 2\pi\left(m+\frac{1}{2}\right)(x-\lambda)}{\sin\pi(x-\lambda)} + 0{,}54\frac{\sin 2\pi\left(m+\frac{1}{2}\right)(x+\lambda)}{\sin\pi(x+\lambda)} + \right.$ $+ 0.23\frac{\sin 2\pi\left(m+\frac{1}{2}\right)\left(x-\lambda-\frac{\pi}{m}\right)}{\sin\pi\left(x-\lambda-\frac{\pi}{m}\right)} + 0.23\frac{\sin 2\pi\left(m+\frac{1}{2}\right)\left(x+\lambda-\frac{\pi}{m}\right)}{\sin\pi\left(x+\lambda-\frac{\pi}{m}\right)} +$ $\left. + 0.23\frac{\sin 2\pi\left(m+\frac{1}{2}\right)\left(x-\lambda-\frac{\pi}{m}\right)}{\sin\pi\left(x-\lambda+\frac{\pi}{m}\right)} + 0.23\frac{\sin 2\pi\left(m+\frac{1}{2}\right)\left(x+\lambda+\frac{\pi}{m}\right)}{\sin\pi\left(x+\lambda+\frac{\pi}{m}\right)}\right\}$	$\frac{C_0}{N} + 2\sum_{j=1}^{m}\frac{C_j}{N}\cos 2\pi j\lambda\left(0.46\cos\frac{\pi j}{m} + 0.54\right)$
5. Bartlett rectangular $\left(\text{where } h = \frac{1}{m}\right)$ $\frac{1}{2m}\left\{\frac{1}{2h}\int_{\lambda-h}^{\lambda+h}\frac{\sin^2\pi m(x-\mu)}{\sin^2\pi(x-\mu)}\,d\mu + \frac{1}{2h}\int_{\lambda-h}^{\lambda+h}\frac{\sin^2\pi m(x+\mu)}{\sin^2\pi(x+\mu)}\,d\mu\right\}$	$\sum_{j=-m}^{m}\frac{C_j}{m}\left(1-\frac{\lvert j\rvert}{m}\right)\frac{\sin 2\pi jh}{2\pi jh}\cos 2\pi jh.$

Table 1 bis

Estimation of the spectral density through the transformation method

Displacement of the estimator $f_N^*(\lambda)$ $b_N(\lambda)$	Dispersion of the estimator $f_N^*(\lambda)$ $D^2[f_N^*(\lambda)]$	Error in quadratic mean $M\,\lvert f_N^*(\lambda) - f(\lambda)\rvert^2$
1. Bartlett $\sim \frac{1}{m} P \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{\sin 2\pi(x-\lambda)}{1-\cos 2\pi(x-\lambda)} f'(x)\,dx$ where P is a constant	$\sim \begin{cases} \frac{2}{3}\frac{m}{N} f^2(\lambda), & \lambda \neq 0 \\ \frac{4}{3}\frac{m}{N} f^2(0), & \lambda = 0 \end{cases}$	$\sim \frac{2}{3}\frac{m}{N} f^2(\lambda) +$ $+ \left[\frac{1}{m} P \int_{-1/2}^{1/2} \frac{\sin 2\pi(x-\lambda)}{1-\cos 2\pi(x-\lambda)} f'(x)\,dx\right]^2$ for $\lambda \neq 0$
2. Rectangular estimation $\sim \frac{f''(\lambda)}{6m^2}$	$\sim \begin{cases} \frac{m}{N} f^2(\lambda), & \lambda \neq 0 \\ \frac{2m}{N} f^2(0), & \lambda = 0 \end{cases}$	$\sim \frac{m}{N} f^2(\lambda) + \left[\frac{f''(\lambda)}{6m^2}\right]^2$, for $\lambda \neq 0$.
3. Truncation of the periodogram $\sim \frac{1}{N} P \int_{-1/2}^{1/2} \frac{\sin 2\pi(x-\lambda)}{1-\cos 2\pi(x-\lambda)} f'(x)\,dx$ where P is a constant	$\sim \begin{cases} \frac{2m}{N} f^2(\lambda), & \lambda \neq 0 \\ \frac{4m}{N} f^2(0), & \lambda = 0 \end{cases}$	$\sim \frac{2m}{N} f^2(\lambda) +$ $+ \left[\frac{1}{N} P \int_{-1/2}^{1/2} \frac{\sin 2\pi(x-\lambda)}{1-\cos 2\pi(x-\lambda)} f'(x)\,dx\right]^2$ for $\lambda \neq 0$
4. Tukey $\sim 0.23 f''(\lambda) \left(\frac{\pi}{m}\right)^2$, if $m^2 = 0(N)$	$\sim \begin{cases} \frac{m}{N} f^2(\lambda)\,(0.46^2 + 2 \cdot 0.56^2), & \lambda \neq 0 \\ \frac{2m}{N} f^2(0)\,(0.46^2 + 2 \cdot 0.56^2), & \lambda = 0 \end{cases}$	$\sim \frac{m}{N} f^2(\lambda)\,(0{,}7748) + \left(0.23\, f''(\lambda)\frac{\pi^2}{m^2}\right)^2$ for $\lambda \neq 0$
5. Bartlett rectangular the same as in the case of Bartlett estimation	$\sim \begin{cases} \frac{2m}{N} f^2(\lambda) \int_0^1 (1-x)^2 \left(\frac{\sin \pi x}{\pi x}\right)^2 dx, & \lambda \neq 0 \\ \frac{4m}{N} f^2(0) \int_0^1 (1-x)^2 \left(\frac{\sin \pi x}{\pi x}\right) dx, & \lambda = 0 \end{cases}$	of type $0(N^{-2/3})$ if $m = kN^{\frac{1}{2}}$

Remark. Relation (2.3) shows that $f_N^*(\lambda)$ is a consistent estimator of $f(\lambda)$, if $W_N(x)$ concentrates its mass in points λ, $-\lambda$, when $N - \infty$, which means that

$$(\forall)\varepsilon > 0: \int\limits_{|x \pm \lambda| < \varepsilon} w_M(x)\, dx \to 1 \quad \text{when } N \to \infty.$$

Theorem 4. Let $w_N(x)$ be even, of square, integrable. Then

$$D^2[f_N^*(\lambda)] \sim \frac{2f^2(\lambda)}{N} \int_{-1/2}^{1/2} w_N^2(x)\, dx. \tag{2.4}$$

An important version of theorem 4 is obtained when it is supposed that $w_N(x) = w(x)$. This case is used when constructing estimators for the spectral function.

From the general study of the spectral density estimators, as well as from the practice it results that *we are not able to improve simultaneously the asympthotical behaviour of the displacement, and to diminish the dispersion.*

Hence, so as to be able to characterize the optimality of an estimator, it was necessary to study the converge in quadratic mean of $f_N^*(\lambda)$ towards $f(\lambda)$. The error in quadratic mean is

$$M|f_N^*(\lambda) - f(\lambda)|^2 = [b_N(\lambda)]^2 + D^2[f_N^*(\lambda)].$$

Remark. A small error in quadratic mean is not the only essential property required from a spectral density estimator. In the case of the Forecasting Problem, for instance, it is required moreover that the estimation be positive with probability one.

b) Algorithms

In the following paragraphs we shall give merely the important steps to be observed by spectral density estimator procedure, as well as the final results, sending for details to [2] and [6]. We shall assume the spectral density $f(\lambda)$, as being sufficiently regular (with a sufficient number of derivatives) and defined on a circle of radius 1/2 ($\lambda = 1/2$ is identified to $\lambda = -1/2$).

According to formula (2.1.2) any spectral density estimation procedure must have three parameters:

(i) a weighting function $w_N(x)$ called "lag window";

(ii) an integer m, called truncation point. In the case when the optimization of the error in quadratic mean is to be effected, and when this is possible, a rational α is taken as parameter, where $m = kN^\alpha$ (K is a constant which achieves the above desiderata);

(iii) an integer q, called spectral calculation number, which gives, when we want to estimate the spectral density over the whole interval, the number of points and the points (frequencies) in which $f_N^*(\lambda)$ is calculated. (it is understood that for a subroutine on the computer, we must take into account, besides those parameters, also the selection volume N, and the selection $x_1, \ldots, x_N$.

(i) Several weighting functions $w_N(x)$ are given into Tables 1 and 1 bis, and there also are studied their properties. All constructed estimators are spectrograph estimators. The weighting functions (3) and (4) are not positive

for any x, and hence the corresponding estimators may take also negative values, and this is why we must take into account the last remark.

Estimation (5), multiplied by 1/m, is a good estimation of the spectral function on interval $\left(\lambda - \frac{\pi}{2m}, \lambda + \frac{\pi}{2m}\right)$.

In all cases $m = 0(N)$ and it tends to infinity when $N \to \infty$. Excepting estimation (2), all estimations in the table are suitable for programming on computer, because they do not need to compute but m moments C_j/N.

(ii) In the majority of spectral analysis works, it is recommended that the spectral density estimation be calculated for several truncation points, as no quantitative or qualitative routines are perfected for interpreting the spectral density obtained for several selections of m.

Parzen recomends in [6] to select every time, by three points of truncation $m_1 < m_2 < m_3$ with m_1 even-number and $N/20 \leqslant m_1 \leqslant N/10$ and $m_2 = 2m_1$ and $m_3 = 2m_2$.

Remarking in Table 1 bis that the estimator dispersions $f_N^*(\lambda)$ depend on factor m/N, we shall be lead to consider as criterion of selecting m the one which diminishes ratio m/N.

Table 2 gives the manner how ratio m/N influences the band in which the spectral density is estimated. (A and B in the Table have the significance from notation $A \leqslant (f_N^*(\lambda) - f(\lambda))/f(\lambda) \leqslant B$. We draw the reader's attention upon the cautious usage of this criterion of selecting m, in order not to increase the displacement, when tempted to diminish the dispersion.

Table 2

Connection between ratio $\frac{m}{N}$ *and the confidence interval*

$\frac{m}{N}$	A	B
0.005	− 0.3	0.4
0.10	− 0.4	0.6
0.20	− 0.5	0.9
0.40	− 0.6	1.5

A third selection mode of the truncation point m is taken into account by Grenander and Rosenblatt in [2]. They intend, in this case, to minimize the error in quadratic mean. Table 3 offers the values of K and α for which the error in quadratic mean from Table 1 is minimum, where $m = KN^{\alpha}$.

(iii) The points (frequencies) in which $f_N^*(\lambda)$ is calculated, when the spectral density is estimated upon the whole interval $(0, \pi)$, have the form $0, \frac{\pi}{q}, \lambda\frac{\pi}{q}, .., \pi$.

Table 3

Optimal values of m for minimizing the error in quadratic mean from table 1

Current number of estimator in Table 1	α, k options where $m = kN^{\alpha}$	Error in quadratic mean
1	$\alpha = \frac{1}{3}$	$0(N^{-2/3})$
2	$\alpha = \frac{1}{5}$ $k = \left(\frac{\lvert f''(\lambda)\rvert}{f(\lambda)}\right)^{2/5} \left(\frac{\pi^2}{3}\right)^{1/5}$	$\sim 5 \cdot 4^{-4/5} (f(\lambda))^{2/5} (\lvert f''(\lambda)\rvert)^{2/5} \left(\frac{\pi^4}{36}\right)^{1/5} N^{-4/5}$
3	There is no optimal α. The selection of a small value is recommended.	$2kN^{\alpha-1} f^2(\lambda)$
4	$\alpha = \frac{1}{5}$ $k = \left[\frac{\lvert f''(\lambda)\rvert}{f(\lambda)}\right]^{2/5} \left(\frac{2116}{7748}\right)^{1/5}$	$\sim 05 \cdot 4^{-4/5} (f(\lambda))^{2/5} (\lvert f''(\lambda)\rvert)^{2/5} (0{,}7748)^{2/5}, \quad (0.23\pi^2)^{1/5} N$
5	$\alpha = \frac{1}{3}$	$0(N^{-2/3})$

We recommend, as a rule, that q, the spectral calculation number, be selected as approximately equal to m. In the case of physical applications, q will be selected so that the frequencies with physical significance be multiple of π/q; in the case of economic applications, q is selected multiple of 12.

The time T for calculating an estimator is approximately equal to the product between m, the truncation point, and q, the spectral calculation number.

Table 4

Estimated calculation times in terms of m and q

m	q	T
40	40	1 sec.
80	80	4 sec.
80	160	8 sec.
200	200	24 sec.
500	200	1 min

Some values of m, q, T, obtained on a IBM computer, are given in Table 4.

III. FILTERING METHOD

a) Theoretical considerations

The filtering method consists in passing a message x_t, $0 < t < T$ through a linear filter $\mathcal{F}$ with response frequency $\varphi_T(y)$ strongly concentrated in point λ, in which the estimation of the spectral density $f(\lambda)$ is required.

The further down constructed estimators are generalizations of thoses from relation (2.1), but they are not spectrograph estimators. They are important for their wide application on analogic computers.

In the following paragraphs we consider, besides the hypotheses from the Introduction, some normal random processes with continuous parameters, noticed in a time interval $0 < t < T$.

Theorem 5 [2]. Let linear filter be, with transient response function $g_T(t)$. Then

$$\alpha_T^* = \frac{1}{T}\int_0^T |z_t|^2 \, dt = \frac{1}{T}\int_0^T\int_0^T x_u x_v \cos(u-v)\lambda \, g_T(t-u) \, g_T(t-v) \, du \, dv \, dt \quad (3.1)$$

where

$$z_t = \mathcal{F} x_t e^{it\lambda} = \int_0^t x_s e^{is\lambda} g_T(t-s) \, ds \quad 0 < t < T$$

is a consistent estimator of the spectral density $f(\lambda)$.

Remark. Transposed on an analogic computer, formula (3.1) consists in mixing message x_t, $0 < t < T$ with a wave $e^{i\lambda\lambda}$, produced by the oscilator with frequency λ, in filtering this mixture with a filter having response frequency $\varphi_T(x)$ and calculating the mean of the output power, given by this filter.

Theorem 6. [2]. The displacement of estimator (3.1) is

$$b_T(\lambda) = 0\left[\sqrt{\int_R (x-\lambda)^2 \, |\varphi_T(x-\lambda)|^2 \, dx}\right] + 0\left(\frac{h^2(T)}{T}\right). \tag{3.2}$$

We define the width of the spectral band $\beta(\alpha_T^*)$.

$$\beta(\alpha_T^*) = \frac{1}{\int_R |\varphi_T(y)|^2 \, dy}. \tag{3.3}$$

Theorem 7. In the conditions of theorem 5 and supposing that

$$\max_{|u| \leqslant \frac{A}{T}} \left| \frac{\int_R |\varphi_T(x)|^2 \, |\varphi_T(x+u)|^2 \, dx}{\int_R |\varphi_T(x)|^4 \, dx} - 1 \right| \to 0 \, (\forall) \, A > 0 \quad \text{and} \quad T \to \infty$$

the dispersion of estimator α_T^* is asymptotical:

$$D^2[\alpha_T^*(\lambda)] \sim \frac{2\pi f^2(\lambda)}{T} \int_R |\varphi_T(y)|^4 \, dy \quad \text{for} \quad \lambda \neq 0 \tag{3.4}$$

and

$$D^2[\alpha_T^*(0)] \sim \frac{4\pi f^2(0)}{T} \int_R |\varphi_T(y)|^4 \, dy \quad \text{for} \quad \lambda = 0.$$

The compromise between the diminution of dispersion and the smoothings and distorsions to which is submitted the process, for obtaining a small displacement (see the analogous problem in 2), is achieved by formula

$$\frac{D^2[\alpha_T^*(\lambda)]}{M^2[\alpha_T^*(\lambda)]} \beta(\alpha_T^*) = \frac{2\pi}{T} \frac{\int_R |\varphi_T(y)|^4 \, dy}{\int_R |\varphi_T(y)|^2 \, dy} = \frac{2\pi}{T} G_1(\varphi_T) \tag{3.5}$$

(dispersion x spectral band width = constant) also called the *indetermination principle*, by analogy with the known Heisenberg principle from statistical mechanics.

When studying the form of estimator (3.1), taking into account that process x_T is normal, and the selections are independent, it results that

$\alpha_T^*(\lambda)$ *is a weighted sum of* χ^2. We introduce the *number of equivalent degrees of freedom* as being

$$EDF[\alpha_T^*(\lambda)] = \frac{2T\beta(\alpha_T^*)}{G_1(\varphi_T)} \tag{3.6}$$

b) Algorithms

In the followin paragraphs we shall give merely the important steps to be observed by a spectral density estimation procedure, as well as the final results, seding for details to [4] and [6].

According to the above presented considerations, any spectral density estimation procedure must have the following parameters:

(i) a response characteristic $\varphi_T(y)$ of the linear filter $\mathcal{F}$;

(ii) an integer n, representing the equivalent number of degrees of freedom $(EDF[\alpha_T^*])$;

(iii) the width of the spectral band $\beta(\alpha_T^*)$; on the basis of the width of the spectral band $\beta(\alpha_T^*)$, we determine m, the truncation point;

(IV) we determine N, the selection volume corresponding to the selected response characteristic φ_T and to the determined truncation point m.

Table 5 gives several response characteristics φ_T, the dispersions, the number of equivalent degrees of freedom — $EDF[\alpha_T^*]$, and the width of the corresponding spectral band $\beta(\alpha_T')$. The selection of the above parameters becomes easy, when using the formulas in this Table.

IV. AUTO-REGRESSIVE ESTIMATION METHOD

The auto-regressive estimation method consists in writing the x_t process as a auto-regression, and in estimating thereafter the spectral density of the thus constructed spectral density.

The auto-regressive estimation method is bassed upon the following theorem:

Theorem 8 [2]. Let x_t be the solution of the stochastic equation with differences

$$a_0 x_{t+p} + a_1 x_{t+p-1} + \ldots + a_p x_t = \eta_t \tag{4.1}$$

where η_t is an uncorrelated process, having zero mean and dispersion σ^2, and $\varphi(z) = \sum_{j=0}^{p} a_j z^{p-j}$ has not real solutions (process x_t *is called auto-regressive process*).

The spectral density of process x_t is

$$f(\lambda) = \frac{\sigma^2}{|\varphi(e^{2\pi i\lambda})|^2}. \tag{4.2}$$

Table 5

Estimation of the spectral density through the filtering method

	Response function $\varphi_m(k)$	Dispersion of estimator $\alpha_m^*(\lambda)$	Number of equivalent degrees of freedom $EDF[\alpha_m^*(\lambda)]$	Spectral band width $B(\alpha_m)$
1. Bartlett (1)	1	$2m/n$	n/m	$1/4\ m$
2. Bartlett (2)	$1 - k/m$	$2m/3n \sim m/n$	$3n/m \sim 2n/m$	$3/4m \sim 1/2\ m$
3. Tukey (1)	$1 - 2a + 2a\ cos\ \frac{\pi k}{m}$ suggested $a = 0.23$	$\frac{2m}{n}(1 - 4a + 6a^2)$	$\frac{n}{m(1 - 4a + 6a^2)}$	$\frac{1}{4n(1 - 4a + 6a^2)}$
4. Tukey (2)	$\frac{1}{2}\left(1 + cos\ \frac{\pi k}{m}\right)$ suggested $a = 0.25$)	$\frac{3m}{4n} \sim \frac{m}{n}$	$\frac{8n}{3m} \sim \frac{2n}{m}$	$\frac{1}{2m}$
5. Parzen (1)	$1 - \frac{k^2}{m^2}$	$\frac{16m}{15n} \sim \frac{m}{n}$	$\frac{15n}{8m} \sim \frac{2n}{m}$	$\frac{1}{2m}$
6. Parzen (2)	$\begin{cases} 1 - \frac{6k^2}{m^2\left(1 - \frac{k}{m}\right)}, & 0 \leqslant k \leqslant \frac{m}{2} \\ 2 - \left(1 - \frac{k}{m}\right)^3, & \frac{m}{2} \leqslant k \leqslant m \end{cases}$	$\frac{0.542\ m}{n} \sim \frac{m}{2n}$	$\frac{3 \cdot 7n}{m} \sim \frac{4n}{m}$	$\frac{1}{m}$
6. Daniell	$\frac{sin\ kh}{kh}$ $k = 0, 1, 2, ..., n$	$\frac{\pi}{nh}$	$\frac{2nh}{\pi}$	$\frac{h}{2\pi}$

Writing x_t only in terms of its past, and taking into account formula (4.1), we obtain

$$x_t = -\frac{1}{a_0}(a_1 x_{t-1} + \ldots + a_p x_{t-p}) + \frac{\eta_t}{a_0}.$$

Denoting by

$$\hat{x}_t = -\frac{1}{a_0}(a_1 x_{t-1} + \ldots + a_p x_{t-p}), \quad \varepsilon(t) = x_t - \hat{x}_t$$

it results that $\varepsilon(t)$ is a weak stationary process with zero mean and dispersion $\sigma^2/|a_0|^2$.

Thereafter, the spectral density estimation algorithm, established through the self-regressive method, consists in the following steps:

(i) determination of the auto-regression coefficients $a_1, \ldots, a_p$ (given a priori as parameters or estimated through a spectral procedure);

(ii) testing the statistical hypothesis that $\varepsilon(t)$ is the white noise (spectral density equal to σ^2);

(iii) if the hypothesis is true, the estimation of the spectral density of process x_t is given by formula (4.2).

V. TESTS AND CONFIDENCE INTERVALS

The test and the confidence intervals, studied in this chapter, are applied for the spectrograph estimators, but they may be easily generalized also for the estimators of the type (4.2).

Theorem 9 [2]:

$$\lim_{m\to\infty} \lim_{N\to\infty} P\left\{\max_k \sqrt{\frac{Nh}{2\pi}} \left| \log\left[\int_{kh}^{(k+1)\pi} \int_{-\pi}^{\pi} I_N(x)\, w_N(x-\lambda)\, dx\, d\lambda / f\left(\left(k+\frac{1}{2}\right)h\right)\right]\right| \leqslant \right.$$

$$\left. \leqslant \Phi^{-1}\left(1-\frac{\alpha}{m}\right)\right\} = e^{-2\alpha}$$

where Φ is the Gaussian probability function, and Φ^{-1} its reverse (reciprocal). On the basis of theorem 9 we can construct the following confidence interval:

$$exp\left\{-\sqrt{\frac{2\pi m}{N}}\,\Phi^{-1}\left(1-\frac{\alpha}{m}\right)\frac{m}{\pi}\int_{\frac{k\pi}{m}}^{\frac{(k+1)\pi}{m}} \int_{-\pi}^{\pi} I_N(x)\, w_N(x-\lambda)\, d\lambda \leqslant\right.$$

$$\left. \leqslant f\left(\left(k+\frac{1}{2}\right)\frac{\pi}{m}\right) \leqslant exp\right\} \sqrt{\frac{2\pi m}{N}}\,\Phi^{-1}\left(1-\frac{\alpha}{m}\right)\frac{m}{\pi}\int_{\frac{k\pi}{m}}^{\frac{(k+1)\pi}{m}} \int_{-\pi}^{\pi} I_N(x)\, w_N(x-\lambda)\, d\lambda$$

for the spectral density $f(\lambda)$ in points $\lambda = \pi/2m,\ 3\pi/2m,\ \ldots,\ (2m-1)\,\pi/2m$.

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METHODES NUMERIQUES DANS L'ESTIMATION DE LA DENSITE SPECTRALE

L'auteur passe en revue les problèmes concernant l'estimation de la densité spectrale des séries dynamiques. Les résultats importants obtenus dans ce domaine ont été synthétisés en trois méthodes: la méthode de la transformation, la méthode du filtrage et celle de l'estimation auto-régressive.

ЧИСЛОВЫЕ МЕТОДЫ ДЛЯ ОЦЕНКИ СПЕКТРАЛЬНОЙ ПЛОТНОСТИ

В работе приведены вопросы, авязанные с оценкой спектральной плотности динамических рядов. Важнейше результаты, полученные в этой области были синтетизированы в трёх методах; метод преобразования, фильтровый метод и метод авторегрессионной оценки.

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