VIVEK PATKAR
P.C. SAXENA
OM PARKASH

Department of Operational Research
University of Delhi — India

Dual Program for a Convex Fractional Function

A dual relation has been developed for a convex fractional function.

1. INTRODUCTION

One of the important properties of duality principle in linear programming is that if one of the pair of dual problems has an optimal solution, then the other also has an optimal solution. Moreover, it is possible to find an optimal solution to the dual once we have determined an optimal solution to the primal, and conversely. The problem, in the dual of which we are interested was studied by Bector [1]. Motivated by the work of Kaska[3] a dual relation is established for a convex fractional programming problem over linear constraint. It is interesting to note that the corresponding dual problem is having objective function to be linear but the constraints are in quadratic form and such problems are encountered in the field of chance — constrained programming [2].

The paper is divided into two sections. First section contains definition of the primal and dual programs and duality theorems are proved in second section.

Section I Primal Problem.

$$\text{Minimize } F(X) = \frac{(C'X)^2}{(D'X)} \tag{1.1}$$

$$\text{Subject to } AX \geqslant b \tag{1.2}$$

$$X \geqslant 0 \tag{1.3}$$

where A is mxn matrix, X, C and D are $n$xl vectors, b is a $m$xl column vector, $D'X$ is strictly positive and $C'X$ is non-negative over the constraint set of primal problem. The dual associated to this primal problem is

Dual Problem. Maximize $G(V, t) = b'V$ (1.4)

subject to $A'V + Dt^2 \leqslant 2Ct$ (1.5)

$V \geqslant 0, t$ (real number) $\geqslant 0$. (1.6)

Let C_P and C_D denote the feasible sets of the primal and dual problems resl pectively, i.e.

$$C_P = [X : X \geqslant 0, \quad AX \geqslant b]$$

$$C_D = [(V, t) : V \geqslant 0, \quad t \geqslant 0, \quad A'V + Dt^2 \leqslant 2Ct].$$

Section II

Theorem 1. If X and (V, t) are feasible solutions for primal and dual problem respectively, then

$$Inf_{C_P} F(X) \geqslant Sup_{C_D} G(V, t).$$

Proof. Adopting the convention [7] that

$$C_P = \varphi, \quad Inf\, F(X) = +\infty$$

$$C_D = \varphi, \quad Sup\, G(V, t) = -\infty.$$

Then it remains to prove the above inequality holds when the constraint set C_P and C_D are nonempty. Now (1.3) and (1.5) gives

$$X'A'V + X'Dt^2 \leqslant 2X'Ct. \tag{2.1}$$

From (1.2) and (1.6) we have

$$X'A'V \geqslant b'V. \tag{2.2}$$

Using (2.2) in (2.1) we get

$$b'V + D'Xt^2 - 2C'Xt \leqslant 0 \tag{2.3}$$

or

$$b'V + [t(D'X)^{1/2} - (C'X)\,(D'X)^{-1/2}]^2 - \frac{(C'X)^2}{(D'X)} \leqslant 0$$

or

$$b'V - \frac{(C'X)^2}{D'X} \leqslant 0$$

or

$$b'V \leqslant \frac{(C'X)^2}{D'X}.$$

Hence

$$\underset{X \in C_P}{Inf} \; F(X) \geq \underset{(V,t) \in C_D}{Sup} \; G(V, t).$$

Corollary: If X is primal feasible and (V, t) is dual feasible such that $F(X) = G(V, t)$, then X is primal optimal and (V, t) is dual optimal.

Theorem 2. If X_0 is the optimal solution of primal problems then there exist $(V_0, t_0) \geq 0$, an optimal solution to the dual problem and satisfies $F(X_0) = G(V_0, t_0)$.

Proof. Since X_0 is primal optimal by Kuhn Tucker theorem [4] there exists $(W_0, Y_0) \geq 0$ such that

$$AX_0 - b \geq 0 \tag{2.4}$$

$$(AX_0 - b)' W_0 = 0 \tag{2.5}$$

$$-A'W_0 + \left[\frac{2(C'X_0)(D'X_0)C - (C'X_0)^2 D}{(D'X_0)^2}\right] - Y_0 = 0 \tag{2.6}$$

$$-X_0'A'W_0 + \left[\frac{2(C'X_0)^2(D'X_0) - (C'X_0)^2(D'X_0)}{(D'X_0)^2}\right] - X_0'Y_0 = 0 \tag{2.7}$$

$$X_0'Y_0 = 0. \tag{2.7 a}$$

Because of $Y_0 \geq 0$, (2.6) is equivalent to

$$A'W_0 + \frac{(C'X_0)^2}{(D'X_0)^2} D \leqslant 2 \frac{C'X_0}{D'X_0} C. \tag{2.8}$$

Setting $t_0 = \dfrac{C'X_0}{D'X_0}$, (2.8) reduces to

$$A'W_0 + Dt_0^2 \leqslant 2Ct_0. \tag{2.9}$$

Thus $(W_0 = V_0, t_0)$ in (2.9) implies the feasibility for dual problem. Now from (2.5) we get

$$X_0'A'W_0 = b'W_0 = b'V_0. \tag{2.10}$$

Using (2.10) with (2.7 a) in (2.7) we have

$$b'V_0 - \frac{(C'X_0)^2(D'X_0)}{(D'X_0)^2} = 0$$

$$b'V_0 = \frac{(C'X_0)^2}{D'X_0}.$$

Hence by theorem 1, proof is completed.

We shall now assume that the constraints of the dual problem (1.5) and (1.6) satisfy a Kuhn Tucker constraint qualification [4].

Theorem 3. If (V_0, t_0) is the dual optimal solution, then there exists X_0 which is primal optimal and satisfies $F(X_0) = G(V_0, t_0)$.

Proof. Since (V_0, t_0) is dual optimal, by Kuhn Tucker Theorem [4] there exists a $(X_1, Y_1) \geqslant 0$ such that

$$-b + AX_1 = Y_1 \geqslant 0 \tag{2.11}$$

$$(b - AX_1)' V_0 = 0, \quad Y_1' V_0 = 0 \tag{2.12}$$

$$-2t_0 D' X_1 + 2C' X_1 = 0 \tag{2.13}$$

$$(A' V_0 + Dt_0^2 - 2Ct_0)' X_1 = 0. \tag{2.14}$$

Is follows from (2.11) that the primal problem is feasible and $X_0 = X_1$. Now

$$\begin{aligned} b' V_0 &= X_1' A' V_0 && \text{[From (2.12)]} \\ &= 2C' X_1 t_0 - t_0^2 D' X_1 && \text{[From (2.14)]} \\ &= 2 \frac{(C' X_1)}{D' X_1} (C' X_1) - \frac{(C' X_1)^2}{(D' X_1)^2} (D' X_1) && \text{[From (2.13)]} \\ &= \frac{(C' X_1)^2}{D' X_1} \end{aligned}$$

i.e. $b' V_0 = \dfrac{(C' X_0)^2}{D' X_0}$.

Thus complete duality relationship between primal and dual problem is established.

REFERENCES

1. BECTOR C. R.: *Programming Problems with Convex Fractional Functions.* Operations Research, no. 16, 1968, p. 383–391.
2. CHARNES A., COOPER W. W.: *Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints.* Operations Research, no. 11, 1963, p. 18–39.
3. KASKA J.: *Duality in Linear Fractional Programming.* Ekonomicko Matematicky, Obzor, no. 4, 1969, p. 442–453.
4. KUHN H. W., TUCKER A. W.: *Nonlinear Programming. Second Berkeley Symposium Proceedings on Mathematical Statistics and Probability,* University of California, Press, Berkeley, California, 1951.
5. MANGASARIAN O. L.: *Nonlinear Programming.* McGraw-Hill Book Co., New York, 1969.
6. VAN DE PANNE C.: *Programming with a Quadratic Constraint,* Management Science, no. 12, 1966, p. 798–815.
7. WOLF P.: *Duality Theorem for Nonlinear Programming.* Quarterly Applied Mathematics, no. 19, 1961, p. 239–244.

14.XII.1977.