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Useful Versions of Entropy with Weights

The authors present in this paper procedures making possible the independence of entropic measure with weights [1] as against the measuring unit of weights (greater than or equal to 1).

The same weight is assigned to all states, in the case of Shannon entropic measure, but in many economy conditions, different efficiencies, hence different weights, correspond to the various states, and this cannot be ignored*.

This is why the authors analysed in [1] various modalities of approaching the problem of entropy with weights, suggesting one modality which seemed to be more suitable.

More precisely, being given an event $\mathcal{E}$ with n possible states $E_1, E_2, \ldots, E_n$ having non-negative weights $h_1, h_2, \ldots, h_n$; if we denote $\ln W_i = h_i (i = 1, 2, \ldots, n)$, then the suggested measure is: $H_p(\mathcal{E}) = \sum_{i=1}^{n} p_i \ln \frac{W_i}{p_i}$.

We supposed, so as to obtain this result, that, as in the case of Shannon's entropy, when two independent events $\mathcal{E}$ and $\mathcal{E}'$ take place, the measure of state $E_i E'_j$ of event $\mathcal{E}\mathcal{E}'$ is equal with the sum of the measures of state E_i of event $\mathcal{E}$, and of state E'_j of event $\mathcal{E}'$, which leads to equation [1]:

$$f_{ij}(xy) = f_i(x) + f_j(x)$$

Particularly, for $x = y = 1$; $f_{ij}(1) = f_i(1) + f_j(1)$ i.e. $h_{ij} = h_i + h'_j$ and as $h_{ij} = \ln W_{ij} \therefore \ln W_{ij} = h_i + h'_j \therefore W_{ij} = e^{h_i + h_j} = e^{h_i} \cdot e^{h_j} = W_i \cdot W'_j$.

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* The efficiency of states in various particular cases appears under various concrete forms, as importance, utility, significance, quality level, pragmatic or semantic value etc. of states. Well determined numerically values are assigned to the various efficiency levels, according to the actual nature of the phenomenon. We suppose that the respective numerical values are given, and they don/t constitute an investigation subject in this study.

When acting so, the problem of selecting a proportionality factor γ of weights h_i, h'_j remains all the same open, as, if $h_{ij} = h_i + h'_j$, then also $\gamma h_{ij} = \gamma h_i + \gamma h'_j$ and hence $\overline{W}_{ij} = e^{\gamma h_i} \cdot e^{\gamma h'_j} = \overline{W}_j \cdot \overline{W}_j$, which makes that the magnitude order of weights $\overline{W}_{ij}$ of the states of event $\mathcal{S}\mathcal{S}'$ be generally completely different from the magnitudes of events $\mathcal{S}$ and $\mathcal{S}'$, which is not natural. As a matter of fact, the selection itself of γ, when the entropy with weights of event $\mathcal{S}$ is considered isolatedly, rises problems generating the considerations from [1]. But it would be more adequate that the weights of states of events $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}\mathcal{S}'$ do not depend on any proportionality factor, in order to remove the imprecision generated by the establishing of weights.

We present further down two procedures which eliminate this difficulty. Let us presume, so as to present things in a more intuitive manner, that, through a certain procedure, we assign to the states $E_1, E_2, \ldots, E_n$ of event $\mathcal{S}$ positive weights $a_1, a_2, \ldots, a_n$, and that the numbering of the states was made in such a way, that the weights be presented in a non-increasing order, i.e. $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$. With a view to having weights not depending on a proportionality factor, it is natural to take, instead of the above mentioned weights, weights having the form: $h'_i = \dfrac{a_i}{a_1 + a_2 + \ldots + a_n} \varphi(n) = h_i \cdot \varphi(n)$ where $\varphi(n)$ is a function depending only on the number of states of event $\mathcal{S}$; we shall come back to this function. Of course $h_1 \geqslant h_2 \geqslant \ldots \geqslant h_n$; $\sum_{i=1}^{n} h_i = 1$; and the same is valid for $h'_1 \geqslant h'_2 \geqslant \ldots \geqslant h'_n$; $\sum_{i=1}^{n} h'_i = \varphi(n)$; h_i, h'_i $(i = 1, 2, \ldots n)$ as it does not depend on any proportionality factor.

If we denote by h_{ij} the weight of state $E_i E'_j$ of event $\mathcal{S}\mathcal{S}'(i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n')$ it would be necessary that $\sum_{i=1}^{n} \sum_{j=1}^{n} h'_{ij} = \varphi(nn')$ because $\mathcal{S}\mathcal{S}'$ is also an event and, if considered isolatedly, the weights of its states would have to satisfy the same requirement as above.

We shall dwell upon two methods for considering the weighths of the states of event $\mathcal{S}\mathcal{S}'$:

1°. We consider as weight of state $E_i E'_j$ of event $\mathcal{S}\mathcal{S}'$, the sum of the weights of states E_i and E'_j of events $\mathcal{S}$ and $\mathcal{S}'$, i.e.

$$h'_{ij} = h'_i + k'_j = h_i \varphi(n) + k_j \varphi(n');$$

2°. We consider as weight of state $E_i E'_j$ the product of the weight of both components states, i.e.

$$h'_{ij} = h'_i \cdot k'_j = h_i k_j \, \varphi(n) \varphi(n').$$

The first method joins oneself with the case studied in [1], the second one requires supplementary considerations.

We begin with the first method. As $\sum_{i=1}^{n}\sum_{j=1}^{n'} h'_{ij} = \varphi(nn')$; it is necessary that $\sum_{i=1}^{n}\sum_{j=1}^{n'}[h_i\varphi(n) + k_j\varphi(n')] = \varphi(nn') \therefore n'\varphi(n) + n\varphi(n') = \varphi(nn') \therefore \frac{\varphi(n)}{n} + \frac{\varphi(n')}{n'} = \frac{\varphi(nn')}{nn'}$.

Denoting $\frac{\varphi(n)}{n} = \psi(n)$, we come to the equation $\psi(n) + \psi(n') = \psi(nn')$ which, in enough general conditions, admits the solution $\psi(n) = c \ln n$ and hence $\varphi(n) = cn \ln n$. In our case $c > 0$ and $h'_i = cn \ln nh_i$.

Passing over to the second method, we remark that here $\sum_{i=1}^{n}\sum_{j=1}^{n'} h'_{ij} = \varphi(n)\,\varphi(n')\sum_{i=1}^{n} h_i \cdot \sum_{j=1}^{n'} k_j = \varphi(n)\cdot\varphi(n')$ so that condition $\sum_{i=1}^{n}\sum_{j=1}^{n'} h'_{ij} = \varphi(nn')$ becomes: $\varphi(n)\cdot\varphi(n') = \varphi(nn')$, whose solution, under quite general conditions is $\varphi(n) = n^{\alpha}$. Hence, in this case $h'_i = n^{\alpha}h_i$.

The entropic neasure with weights found in [1], as well as some properties of the composed events depended upon the fact that the weight of state $E_iE'_j$ is equal with the sum of weights of states E_i and E'_j, a requirement satisfied above only in case of method 1°.

What happens in case of method 2°? Which is the entropic measure, which are the composition laws? What equation would we consider in this case?

As f_i, f_j, f_{ij} are decreasing functions of argument x; $x \in [0, 1]$, the minimal values being $f_i(1) = h'_i$; $f_j(1) = k'_j$; $f_{ij}(1) = h'_{ij}$, the equation to be considered is as in [1]:

$$f_{ij}(xy) - h'_{ij} = [f_i\cdot(x) - h'_i] + [f_j(y) - k'_j]$$

with the only difference that here h'_{ij} is no longer equal with $h'_i + k'_j$ but with $h'_i \cdot k'_j$. Taking into the above equation $y = 1$ and $x = 1$, respectively, it results:

$$f_{ij}(x) - h'_{ij} = f_i(x) - h'_i; \quad f_{ij}(y) - h'_{ij} = f_j(y) - k'_j,$$

thus, equation may be also written

$$f_{ij}(xy) - h'_{ij} = [f_{ij}(x) - h'_{ij}] + [f_{ij}(y) - h'_{ij}].$$

Denoting $f_{ij}(x) - h'_{ij} = F(x)$, the equation becomes:

$$F(xy) = F(x) + F(y).$$

where F is the decreasing function of argument, and $F(1) = 0$. We are in the situation of the equation considered in [2], whose solution — to say nothing of a multiplicative constant — is $F(x) = \ln\frac{1}{x}$. Consequently: $f_i(x) - h'_i = \ln\frac{1}{x}$; $f_i(x) = \ln\frac{1}{x} + h'_i$ and denoting $h'_i = \ln W_i$ (so $W_i = e^{h'_i}$), the solu-

tion may be written $f_i(x) = \ln \frac{W_i}{x}$. We must keep in mind that, for obtaining this result, we did not use the way h'_{ij} is connected to h'_i and k'_j.

Reasoning as in [1], we can see that the entropic measure has the same form as there: $\sum_{i=1}^{n} p_i \ln \frac{W_i}{p_i}$.

Now, one more thing. We saw that, within method 1°: $h'_i = cn \ln nh_i$ $(c > 0)$, and within method 2°: $h'_i = n^{\alpha} h_i$, and $h'_1 \geqslant h'_2 \geqslant \ldots \geqslant h'_n$; $\sum_{i=1}^{n} h_i = 1$. Consequently, in both casses $1 < W_1 \leqslant W_2 \leqslant \cdots \leqslant W_n$ and $W_1 . W_2 \ldots W_n = e^{h'_1} \cdot e^{h'_2} \ldots e^{h'_n} = e^{h'_1 + h'_2 + \cdots + h'_n}$ is equal with $e^{cn \ln n}$, in the first method's case, and with $e^{n^{\alpha}}$ in the second method's case. Acting as showed before, the initial weights h_i do not depend any longer on any proportionality factor. But the obtained solutions present two drawbacks, which cannot be ignored. We cannot become as a particular case, Shannon's entropy, because every $W_i > 1$ and not even in the case of the concentration in a single state can we obtain entropy 0, when the state is optimal (with maximal efficiency), and no entropy the greater so as the state is less satisfactory.

This is why we shall introduce a certain corrective within the above weights system, and a more complete study shall be done, taking into account the new weights.

2°. We shall now consider anew, successively, both methods, but adequately modified, so as to eliminate the above mentioned drawbacks.

We start again from the initial weights $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_{n'}$ of the states of events $\mathcal{E}, \mathcal{E}'(a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0; b_1 \geqslant b_2 \geqslant \ldots \geqslant b > 0)$ but we shall use as starting point, instead of weights $h_i = \frac{a_i}{a_1 + a_2 + \ldots + a_n}$; $k_j = \frac{b_i}{b_1 + b_2 + \ldots + b_{n'}}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n')$ the auxiliary weights $\bar{h}_i = h_1 - h_i$; $\bar{k}_j = k_1 - k_j$. In this way $\bar{h}_1 = 0$; $\bar{h}_1 \leqslant \bar{h}_2 \leqslant \ldots \leqslant \bar{h}_n$ and the same is valid for $\bar{k}_1 = 0$; $\bar{k}_1 \leqslant \bar{k}_2 \leqslant \ldots \leqslant \bar{k}$. If $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b'_n$, respectively, then $\bar{h}_1 = \bar{h}_2 = \ldots = \bar{h}_n = 0$, and $\bar{k}_1 = \bar{k}_2 = \ldots = \bar{k}_n = 0$, respectively.

Excepting such situations, hence, if at least $\bar{h}_n \neq 0$, or $\bar{k}_n \neq 0$, respectively, we shall consider instead of weights $\bar{h}_i, \bar{k}_j$ from the previous point, not the auxiliary weights $\bar{h}_i, \bar{k}_j$, because $\sum_{i=1}^{n} \bar{h}_i = nh_1 - 1$; $\sum_{j=1}^{n'} \bar{k}_j = n'k_1 - 1$ are generaly differing from 1, but the weights:

$$d_i = \frac{\bar{h}_i}{\bar{h}_1 + \bar{h}_2 + \ldots + \bar{h}_n} = \frac{a_1 - a_i}{na_1 - (a_1 + a_2 + \ldots + a_n)};$$

$$e_j = \frac{\bar{k}_j}{\bar{k}_1 + \bar{k}_2 + \ldots \bar{k}_{n'}} = \frac{b_1 - b_j}{nb_1 - (b_1 + b_n + \ldots b_{n'})}.$$

In this way,

$$\sum_{i=1}^{n} d_i = 1;\ \sum_{j=1}^{n'} e_j = 1;\ 0 = d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n;\ 0 = e_1 \leqslant e_2 \leqslant \ldots \leqslant e'_n;$$

$d_n, e'_n \neq 0$.

Of course, if $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b_n$, respectively, the above formulas make no sense. We shall take through convention, in such situations, $d_1 = d_2 = \ldots = d_n = 0$ and $e_1 = e_2 = \ldots e_n = 0$, respectively.

As we did not use, at the preceding point, the weights h_i, k_j, but the weights $h'_i = h_i\varphi(n)$; $k'_j = k_j\varphi(n')$, we shall do the same now, when we shall not use the weights d_i, e_i, but the weights $d'_i = d_i\varphi(n)$; $e'_j = e_j\varphi(n')$.

Within the method 1°, things occur quite identically as to the previous point, if d_n, $e_n \neq 0$. One takes $d'_{ij} = d'_i + e'_j$, and we deduce as we did there that $\varphi(n) = cn \ln n$ $(c > 0)$. If $d_n = 0$, $e_n = 0$ we find ourselves in the case of Shannon's entropic measure. If only one of those is equal to 0, e.g. $e_n = 0$, which implies $d'_{ij} = d'_i \therefore \sum_{i=1}^{n}\sum_{j=1}^{n'} d_{ij} = \sum_{j=1}^{n'}\sum_{i=1}^{n} d'_i \therefore \varphi(nn') = n'\varphi(n)$ hence in such a situation $\varphi(n) = cn$ $(c > 0)$, but such a case is an exception.

Within the method 2°, the analogy goes not so far as with 1°. Here the weight of state $E_iE'_j$ of event $\mathcal{E}\mathcal{E}'$ is obtained starting from the auxiliary weight $h_{ij} = h_{11} - h_{ij} = h_1, K_1 - h_ik_j$. We must remark that $\bar{h}_{nn'} = 0$ only then, when equalities $h_1 = h_n$; $k_1 = k'_n$ take place cocomitantly, hence ony when equalities: $a_1 = a_2 = \ldots = a_n$; $b_1 = b_2 = \ldots = b_{n'}$.

We define now d_{ij} similarly with d_i, e_j, namely: if $\bar{h}_{nn'} > 0$; $d_{ij} = \dfrac{\bar{h}_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n'}\bar{h}_{ij}} = \dfrac{h_1k_1 - h_ik_j}{nn'k_1h_1 - 1}$, and if $\bar{h}_{nn'} = 0$, $d_{ij}=0$, and similarly $d'_{ij}=d_{ij}\varphi\,(nn')$.

As. $d_i = \dfrac{h_1 - h_i}{nh_1 - 1}$; $e_j = \dfrac{k_1 - k'_j}{n'K_1 - 1}$ hence $h_i = h_1 - (nh_1 - 1)d_i$; $k_j = k_1 -$ $- (n'k_1 - 1)e_j$ we see that: $d_{ij} = \dfrac{h_1k_1 - [h_1 - (nh_1 - 1)d_i]\cdot[K_1 - (n'k_1 - 1)e_j]}{nn'h_1k_1 - 1} =$

$$= \frac{(nh_1 - 1)k_1}{nn'k_1h_1 - 1}\,d_i + \frac{(n'k_1 - 1)h_1}{nn'h_1k_1 - 1}\,e_j - \frac{(nh_1 - 1)\cdot(n'k_1 - 1)}{nn'h_1k_1 - 1}\,d_ie_j.$$

This is the relation linking d_{ij} to d_i and e_j. Would we take $d'_i = d_i\varphi(n,h_1)$; $e'_j = e_j\varphi(n',\ k_1)$; $d'_{ij} = d_{ij}\varphi(nn'h_1k_1)$, and $\varphi(nn'h_1k_1) = \varphi(nh_1)\cdot\varphi(n'k_1)$ then the above relation would become:

$$d'_{ij} = \frac{nh_1 - 1}{nn'h_1k_1 - 1}\,k'_1d'_i + \frac{n'k_1 - 1}{nn'h_1k_1 - 1}\,h'_1e'_j - \frac{(nh_1 - 1)\cdot(n'k_1 - 1)}{nn'h_1k_1 - 1}\,d'_i\cdot e'_j.$$

But if we would take $\varphi(n,h_1) = n^{\alpha}\cdot h_1^{\beta}$ and hence $\varphi(n',k_1) = n'^{\alpha}\cdot k_1^{\beta}$; $\varphi(nn'\ h_1k_1) = (nn)^{\alpha}\cdot(h_1k_1)^{\beta}$; α, β real numbers, the functional equation concerning φ would, be satisfied, hence the above relation too.

But it is more simple and more convenient to use directly the initial formulas:

$$d'_i = \frac{h_1 - h_i}{nh_1 - 1} n^\alpha h_1^\beta ; \; e'_i = \frac{k_1 - k_i}{nk_1 - 1} n'^\alpha K_1^\beta ; \; d'_{ij} = \frac{h_1 k_1 - h_i k_j}{nn' h_1 k_1 - 1} (nn')^\alpha \cdot (h_1 k_1)^\beta ,$$

the passing over to the concomitant consideration of several states being then immediate.

Let us retain now that, in both methods, if $h_n > 0$; $0 = d'_1 \leqslant d'_2 \leqslant \ldots \leqslant d'_n$ and $\sum_{i=1}^{n} d_i = cn \ln n$ in the first method and with $n^\alpha h_1^\beta$ in the second one, constants c, α, β having to be selected according to necessities: for instance $c = 1$; $\alpha = 1$; $\beta = 0$.

From what we showed to point 1, it results that, denoting in both cases $\ln W_i = d'_i$; $\ln W'_j = e'_j$; $\ln W_{ij} = d'_{ij}$, the entropies with the new weights of the events $\mathfrak{E}, \mathfrak{E}'$, $\mathfrak{E}\mathfrak{E}'$ ($\mathfrak{E}$ and $\mathfrak{E}'$ independent) are:

$$H_p(\mathfrak{E}) = \sum_{i=1}^{n} p_i \ln \frac{W_i}{p_i}; \quad H_p(\mathfrak{E}') = \sum_{j=1}^{n} p'_i ln \frac{W'_j}{p_j}; \quad H_p(\mathfrak{E}\mathfrak{E}') = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \ln \frac{W_{ij}}{p_i p_j} .$$

What can we say about the weights w_i? As $W_i = {}^{d'i}$, we can see that $W_1 = 1$, and that $W_i \leqslant W_2 \leqslant \ldots \leqslant W_n$ and $W_1 \cdot W_2 \ldots W_n = e^{d'_1} \cdot e^{d'_2} \ldots e^{d'_n} = e^{d'_1 + d'_2 + \ldots + d'_n} = e^{cn \ln nn} = n^{cn}$ with the first method, and with $e^{n^{\beta}_{1}}$ with the second method.

3. The entropic measure $H(\mathfrak{E}) = \sum_{i=1}^{n} p_i \ln \frac{W_i}{p_i}$ contains as a particular case, Shannon's entropy (if $h_n = 0$). The case wherein $h_n > 0$ presents interest. Let us concentrate upon the probabilities of apparition of states E_1, E_2, ..., E_n, hence upon $p_1, p_2, \ldots, p_n$. If only E_1 (optimal state) takes place, then $p_1 = 1$, and the entropic measure is $\ln W_1 = \ln 1 = 0$, as desired. If only state E_i takes place, then $p_i = 1$ and the entropic measure is $\ln W_i = d'_i$. The entropic measure is the greater so as the state is more unfavorable.

Let us not forget that, in the main, the entropy with weights is a measure of the diversifying level. As we exhibited in [1] the maximal diversification takes place when $p_i = \frac{W_i}{W_1 + W_2 + \ldots + W_n}$ for every i ($i = 1, 2, \ldots, n$) and the corresponding (maximal) entropic measure is $\ln (W_1 + W_2 + \ldots W_n)$.

The following question naturally arises: how great may become $W_2 + W_3 + \ldots + W_n (W_1 = 1)$ when $W_2, W_3, \ldots, W_n$ varies in such a way that $1 \leqslant W_2 \leqslant W_3 \leqslant \ldots \leqslant W_n$ and $W_2 \cdot W_3 \ldots W_n = n^{cn}$ or $e^{n^\alpha h_1^\beta}$, respectively. We denote for convenience $W_2 \cdot W_3 \ldots W_n = \alpha$; $\alpha > 1$. So as to be able to answer the above question, we shall rely upon a simple remark. If $1 \leqslant x < y$ and $xy = a$, the maximum of $(x + y)$ takes place when $x = 1$, $y = a$.

As a matter of fact, as $(x+y)^2=(y-x)^2+4xy=(y-x)^2+4a$, the maximum of $(x+y)$ takes place when also $(y-x)$ is maximal, hence when $x=1$, $y=a$, because otherwise $(y-x)$ takes smaller values.

Let us find now the maximum of $W_2+W_3+\ldots+W_n$. As $W_2\cdot W_3\ldots W_n=\alpha$, it results $W_n=\dfrac{\alpha}{W_2\cdot W_3\ldots W_{n-1}}$, so that the sum may be also written $W_2+W_3+\ldots+W_{n-1}+\dfrac{\alpha}{W_2\cdot W_3\ldots W_{n-1}}$. Let us establish provisionally $W_3, W_4, \ldots, W_{n-1}$. The problem of maximum is put then for $W_2+\dfrac{\alpha}{W_2\cdot W_3\ldots W_{n-1}}$. As the product of the terms is $\dfrac{\alpha}{W_3\cdot W_4\ldots W_{n-1}}$ the maximum of the sum will take place when $W_2=1$. But then the problem of maximum arises further for $W_3+W_4+\ldots+W_{n-1}+\dfrac{\alpha}{W_3\cdot W_4\ldots W_{n-1}}$

Determining $W_4, \ldots, W_{n-1}$ it results, as above, that $W_3=1$.

Going further on in the same way, we see that $W_2=W_3=\ldots W_{n-1}=1$ and $W_n=\alpha$. Therefore, the maximum of $W_1+W_2+\ldots+W_n$ is $\underbrace{1+1+\ldots+1}_{n-1 \text{ times}}+\alpha=n-1+\alpha$ and the maximum of $\ln(W_1+W_2+\ldots+W_n)$ is $\ln(n-1+\alpha)$, the maximum being reached when $p_1=p_2=\ldots=p_{n-1}=\dfrac{1}{n-1+\alpha}$.

If the weights would be the above ones and would take place only in the most unfavourable state, hence if $p_n=1$, the entropic measure would be $\ln\alpha$, smaller than the above one, as is natural. More precisely, within the first method, the maximal entropy is $\ln(n-1+n^{cn})$, and the entropy of the most unfavourable state $\ln n^{cn}$, while within the second method; $\ln(n-1+e^{n^\alpha h_1^\beta})$, or $\ln e^{n^\alpha h_1^\beta}=n^\alpha h_1^\beta$, respectively.

As is known, Shannon's entropy reaches its maximum when $p_1=p_2=\ldots=p_n=\dfrac{1}{n}$. Let use se what happens to the entropy with weights in this contingency.

$$\text{As } H_p(\mathcal{E})=\sum_{i=1}^{n}p_i\ln\frac{W_i}{p_i}=\sum_{i=1}^{n}p_i\ln W_i+\sum_{i=1}^{n}p_i\ln\frac{1}{p_i}=\sum_{i=1}^{n}p_i'd_i+\sum_{i=1}^{n}p_i\ln\frac{1}{p};$$

the first component representing the weighting effect, the second one being Shannon's entropy, we see that when $p_1=p_2=\ldots=p_n=\dfrac{1}{n}$, the first component is equal with $\dfrac{d_1'+d_2'+\ldots+d_n'}{n}$ and the second one with $\ln n$. Within the first method, the contribution of the first component is $c\ln n$, and in the case of the second one $n^{\alpha-1}h_1^\beta$.

If we would consider as natural that, in this situation, the weighting effect be approximately the same as that of diversification, we would have to take $c = 1$ in the first eventuality, while in the second one, if we would suppose in addition that all states have approximately the same weight, hence that h_1 is near $\frac{1}{n}$, it would be necessary that the first component be small enough as against the second one. In other words expressed, it would be necessary that $n^{\alpha-1}\left(\frac{1}{n}\right)^{\beta} = n^{\alpha-1-\beta}$ has an exponent less than 0, i.e. $\alpha - 1 - \beta < 0$. For instance, we could take $\alpha = \beta = 1$; $\alpha = \beta = 2$, etc., but a reasonable selection would imply also other considerations.

Let us now go further to the concomitant achievement of several events. As within the first method $d'_{ij} = d'_i + e'_j$; $W_{ij} = W_i \cdot W_j$, if $\mathcal{E}$ and $\mathcal{E}'$ are independent, the situation is the same as in [1], so that: $H_p(\mathcal{E}\mathcal{E}') = H_p(\mathcal{E}) + H_p(\mathcal{E}')$ the additivity being maintained also when several states are involved.

The case when events $\mathcal{E}$ and $\mathcal{E}'$ are not independent requires a certain attention and the repetition of certain known things. Let us consider the state $E_iE'_j$ of event $\mathcal{E}\mathcal{E}'$, and let p_{ij} be the probability of appearance of this state. We can say that state $E_iE'_j$ takes place when there appears state E_i and then state E'_j, knowing that state E_i appeared previously, hence when states E_i and E'_j/E_i appeared. The probability of appearance of the first state is p_i, and of the second one $p_{j/i}$. If we know that state E_i appeared and that afterwards follows event $\mathcal{E}'$, more precisely event $\mathcal{E}'/E_i$, then one of the states will appear surely, hence $\sum_{j=1}^{n'} p_{i/j} = 1$. We must now remark that we must make a difference between the weights of the states of event $\mathcal{E}'$, unlinked by any previous conditions, and those corresponding to the event $\mathcal{E}'$, taken into account only then, where previously took place state E_i of event $\mathcal{E}$. The last event must be denoted, as a matter of fact, by $\mathcal{E}'/E_i$, and its states by $E'_j/E_i (j = 1, 2, \ldots, n')$. States E'_j/E_i are essentially E'_j states, and the preference order must be naturally considered in the same way as states E'_j, but the weights will be generally different, as they are influenced by the previous appearance of state E_i. We denoted the initial weight of states $\mathcal{E}'$ by $b_1, b_2, \ldots, b_{n'}$; $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{n'} > 0$. We shall denote the initial weights of states E'_j/E_i by $b_{1/i}, b_{2/i}, \ldots, b_{n'/i}$; $b_{1/i} \geqslant \ldots \geqslant b_{n'/i} > 0$. While the normal weights of event $\mathcal{E}'$ are $k_j = \frac{b_j}{b_1 + b_2 + \ldots + b_n}$ those of event $\mathcal{E}'/E_i$ are $k_{j/i} = \frac{b_{j/i}}{b_{1/i} + b_{2/i} + \ldots + b_{n'/i}}$.

Further, instead of $\overline{K}_j$ there appears $\overline{K}_{j/i} = K_{1/i} - K_{j/i}$, and instead of e_j; $e_{j/i} = \frac{\overline{K}_{j/i}}{\overline{K}_{1/i} + \overline{K}_{2/i} + \ldots \overline{K}_{j/n}}$. Finally $d'_{ij} = d'_i + e'_{j/i}$ where $e'_{j/i} =$

$= cn \ln ne_{j/i}$. Denoting $W_{j/i} = e^{e'_{j/i}}$, we see that $W_{ij} = W_i \cdot W_{j/i}$. Further on all is the same as in [1], i.e. $H_p(\mathcal{E}\mathcal{E}') = H_p(\mathcal{E}) + H_p(\mathcal{E}'/\mathcal{E})$ where $H_p(\mathcal{E}'/\mathcal{E}) = \sum_{i=1}^{n} p_i H_p(\mathcal{E}'/E_i)$.

Things take place alltogether different within the second method. Here we have to consider separatedly both components of the entropy with weights, both when events $\mathcal{E}$ and $\mathcal{E}'$ are independent, and when they are dependent.

We begin by mentioning that $H_p(\mathcal{E}) = \sum_{i=1}^{n} d'_i p_i + H(\mathcal{E})$; $H_p(\mathcal{E}') = \sum_{j=1}^{n'} e'_i p'_j + H(\mathcal{E}')$; $H_p(\mathcal{E}'/E_i) = \sum_{j=1}^{n'} e'_{j/i} p'_{j/j} + H(\mathcal{E}'/E_i)$; $H_p(\mathcal{E}\mathcal{E}') = \sum_{i=1}^{n}\sum_{j=1}^{n'} d'_{ij} p_{ij} + H(\mathcal{E}\mathcal{E}')$.

In the case of independence $d'_{ij} = \frac{h_1 k_1 - h_i k_j}{nn' h_1 k_1 - 1} (nn')^{\alpha} \cdot (h_1 k_1)^{\beta}$; $p_{ij} = p_i p_j$; $H(\mathcal{E}\mathcal{E}') = H(\mathcal{E}) + H(\mathcal{E}')$, so that $H_p(\mathcal{E}\mathcal{E}') = \frac{(nn')^{\alpha} (h_1 k_1)^{\beta}}{nn' h_1 k_1 - 1} \left[h_1 k_1 - \sum_{i=1}^{n} h_i p_i \cdot \sum_{j=1}^{n'} k_j p'_j \right] + H(\mathcal{E}) + H(\mathcal{E}')$.

If we would have tried to express $H_p(\mathcal{E}\mathcal{E}')$ with the help of $H_p(\mathcal{E})$ and $H_p(\mathcal{E}')$, the expression would have been more complicated and would have been more difficulty suitable for more states.

In the case of dependence $d'_{ij} = \frac{h_1 k_{\ i} - h_i k_{j/i}}{(nn' h_1 K_{1/i} - 1} (nn')^{\alpha} (h_1 k_{1/i})^{\beta}$; $p_{ij} = p_i \cdot p_{j/i}$ and $H(\mathcal{E}\mathcal{E}') = H(\mathcal{E}) + H(\mathcal{E}'/\mathcal{E})$, so that $H_p(\mathcal{E}\mathcal{E}') = (nn')^{\alpha} h_1^{\beta} \times \left[h_1 \sum_{i=j}^{n} \frac{K_{1/i}^{\beta+1}}{nn' h_1 K_{1/i} - 1} p_i - \sum_{i=1}^{n} \frac{h_i K_{1/i}^{\beta}}{nn' h_i K_{1/i} - 1} p'_i \sum_{j=1}^{n'} k_{j/i} P_{j/i} \right] + H(\mathcal{E}) + H(\mathcal{E}'/\mathcal{E})$.

Certainly, if $\mathcal{E}$ and $\mathcal{E}'$ are dependent, $k_{1/1} = k_{1/2} = \ldots k_{1/n} = k$; $P_{j/i} = p_j (i = 1, 2, \ldots, n)$, and the above expression becomes that of the independence case.

From the two methods presented here, the first one seems to be more natural and presents more similitudes with Shannon's entropy, the second one allows to take into account much more various situations, due to the two parameters α and β, intervening there.*

4. Finally, we present a simple exemple refferring to the first modality represented by graphs, corresponding to 2 states with different weight.

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* If the efficiency of states expresses their semantic or pragmatic value, the measure proposed by us may be interpreted as a measure of semantic or pragmatic information, respectively.

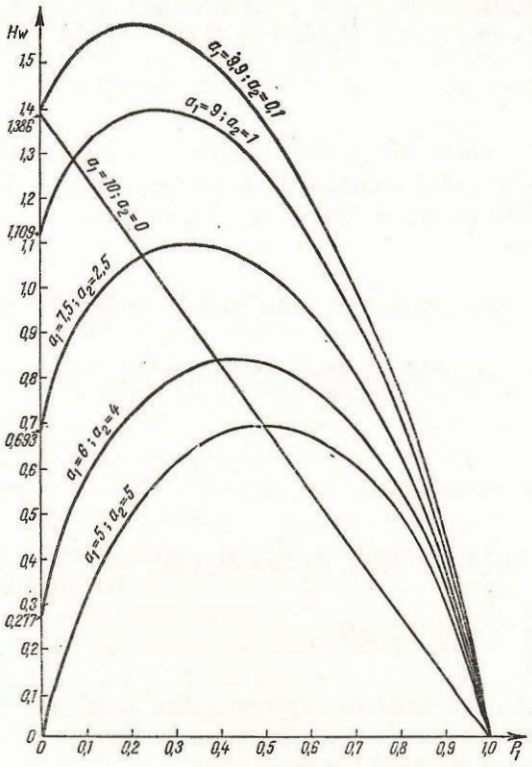
Hw
1,5
1,4
1,386
1,3
1,2
1,109
1,1
1,0
0,9
0,8
0,7
0,693
0,6
0,5
0,4
0,3
0,277
0,2
0,1
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
P1
a1=9,9; a2=0,1
a1=9; a2=1
a1=10; a2=0
a1=7,5; a2=2,5
a1=6; a2=4
a1=5; a2=5

Fig. 1

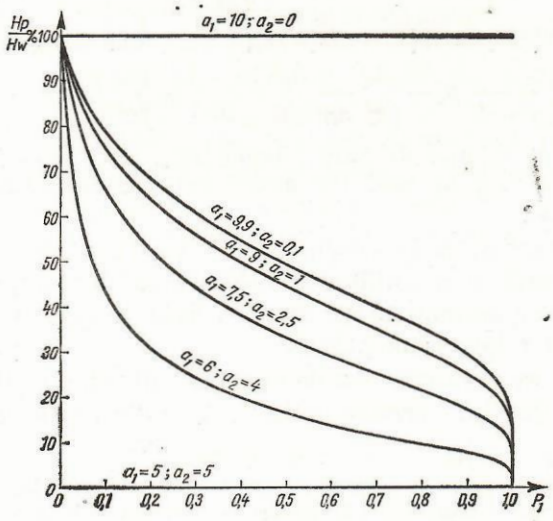
Hp/Hw %
100
90
80
70
60
50
40
30
20
10
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
P1
a1=10; a2=0
a1=9,9; a2=0,1
a1=9; a2=1
a1=7,5; a2=2,5
a1=6; a2=4
a1=5; a2=5

Fig. 2

The initial weights are a_1 and a_2, where from the weights $h_i = \frac{a_1 - a_i}{\sum a_i}$; $i=1,2$, are deduced, and then the entropy with weight like in modality 1, which finally is equal to

$$H_w = \underbrace{\sum p_i \ln \frac{1}{p_i}}_{H_s} + \underbrace{n \ln n \frac{\sum_i (a_1 - a_i)}{\sum_i a_i} p_i}_{H_p}$$

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VARIANTES UTILES DE L'ENTROPIE A POIDS

Les auteurs présentent des méthodologies qui peuvent rendre possible l'interdépendance entre la mesure entropique à poids [1] et l'unité de mesure des poids (plus grande ou égale à 1).

ПОЛЕЗНЫЕ ВАРИАНТЫ ЭНТРОПИИ С ВЕСАМИ

Авторы статьи приводят методологии, способные сделать возможной взаимозависимость энтропической меры с весами [1] и единицой измерения весов (больше или равно единице).

23. 09. 1979