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An Optimal Proceeding for Solving Large-Scale Problems: Vertical Decomposition

The paper presents a proceeding for decomposition of large-scale systems into products, assemblies, subassemblies, half-finished products and complex parts using the method of vertical decomposition.

The decomposition of large-scale problems into problems of relatively small size makes possible their solving using the current computation configurations and the existing optimization packages.

The method of vertical decomposition allows the optimal co-ordination of large-scale systems both by means of constraints and of performance functions.

In the end there are presented results obtained for a system whose dynamic model for production planning and scheduling has been optimally decomposed and co-ordinated.

1. INTRODUCTION

Industrial production processes generally involve manufacturing of complex products, which require a great number of parts and operations. At the same time, an important amount of raw materials and materials, processed on various equipment, participate in the achievement of the respective parts and operations (in the presence of qualified manpower).

The succession sequence of operations which are executed on various equipment, the intermediate stocks gathering between operations, groups of machines, workshops and departments generate an extremely great number of states (stocks) and controls (operations).

The modelling of the respective system becomes necessary in view of increasing the economic efficiency of a single plant, aggregated plants or ministries, in the conditions of the existence of problems such as:

— raw material resources are limited

— circulating means are normed

— equipment should be fully used (co-operation)

— complying with the clauses in conventions and home and foreign contracts

— the use of skilled manpower for each operation, according to operation sheet provisions, in view of obtaining competitive products (as regards quality and price).

The modelling of these systems, aiming at a high economic efficiency, in view of production management and planning, will lead to large scale models, which can not be solved by means of the existing optimization programme and computing-equipment configurations. That is why, in order to solve these large-scale dynamic models, we shall have to resort to decomposition and co-ordination techniques, and to computation methods which will make possible the attainment of optimum results (equal to the global optimum).

Starting from the dynamic model [1], [2], [3], [4], [9] which presents the evolution of the production process, we show a form of the model where:

— the evolution in time of production process is modelled

$$x(k+1) = Ax(k) + Bu(k) + Cf(k) \tag{1}$$

— the constraints imposed on the dynamics of the system which model the conditions of material continuity of the technological flow and the conditions under which all the orders are achieved within the capacity of the available equipment

$$Px(k) + Qu(k) \leqslant R(k) \tag{2}$$

— the initial state of the system

$$x(0) = x_0 \tag{3}$$

— conditions of non-negativity

$$x(k) \geqslant 0 \text{ and } u(k) \geqslant o \text{ for } k = 0, 1, \ldots, N-1 \tag{4}$$

— the objective function

$$J = \max_{x,u}(\min)\left\{\sum_{k=0}^{N-1}[c_1x(k) + c_2u(k)]\right\} \tag{5}$$

by its solving a production plan (production programme) under the form of a set of orders and states: $x(0)$, $u(0)$, $x(1)$, $u(1)$, . . ., $x(N-1)$, $u(N-1)$; $x(N)$ is obtained.

As production planning and scheduling are large-scale problems, decomposition becomes necessary, owing to the limited solving possibilities of both hardware and software.

The decomposition of large-scale systems can be made:

— technologically (horizontal decomposition) [5], [6], [7], [8], by departments, workshops, equipment groups;

— by products, assemblies, subassemblies, half-finished products, complex parts (vertical decomposition) [7], [8].

In the following we shall analyse vertical decomposition [7], [8], as we wish the sum of partial optimal solutions to be equal to the global optimum, that is maximum economic efficiency.

2. A PROCEDURE OF VERTICAL DECOMPOSITION

In order to solve large-scale systems we proceed to their decomposition. The decomposition of the large-scale problem will lead to subproblems of reasonable size, which can be solved by existing computation means and mathematical techniques (optimization packages). By vertical decomposition (by products, subproducts, assemblies, subassemblies, groups of operations leading to a complex part), subproblems of the large-scale problem are obtained. These subproblems can be:

— main problems, which have solving priority;

— secondary problems, which will be solved after obtaining results for all the main subproblems.

In solving a system we start from its (known) outputs. Considering that we have the dynamic model of the system [1], [2], [3], [4], [9], which we want to solve, we shall start from the known — one or several — outputs of the system. By decomposing the large-scale problem using the method of vertical decomposition [7], [8], we shall obtain main subproblems, whose inputs and outputs are known, and secondary problems, whose outputs will be known by a previous solving of the main subproblems a.s.o.

Considering the dynamic model of the system, we notice that we have to solve a graph.

In order to start solving the system, we shall examine the graph, starting from the (known) outputs. We shall determine the principal and secondary subproblems (considering the product operation sheets, the supply plan, the delivery plan and the available capacity for existing equipment as known). The first are obtained by determining the largest branch (branches) of the graph. We consider the graph branches to be vertical, hence the decomposition is considered also vertical.

The longest branch (branches) of the dynamic model, is solved the first.

If the dynamic model has several such branches (equal in length, number of operations and states), we shall start from the system's outputs, which are known, and we shall select the vertical branch with the highest number of links (nodes). If there are several equal vertical branches (equal in number of elements, controls and links nodes) and all of them have outputs they will be solved one after other, the first, the second a.s.o., up to the last.

The same procedure will be used for secondary subproblems, whose outputs are determined by solving the main subproblems.

Solving large-scale systems by vertical decomposition leads to optimal solutions for each principal and secondary subproblem, and their summing up will be equal to the optimal solution obtained by solving globally the system.

3. AN EXAMPLE: THE DYNAMIC MODEL OF A SYSTEM DECOMPOSED AND CO-ORDINATED BY VERTICAL DECOMPOSITION

Starting from the system modelled in [7], [8], for production management, planning and scheduling:

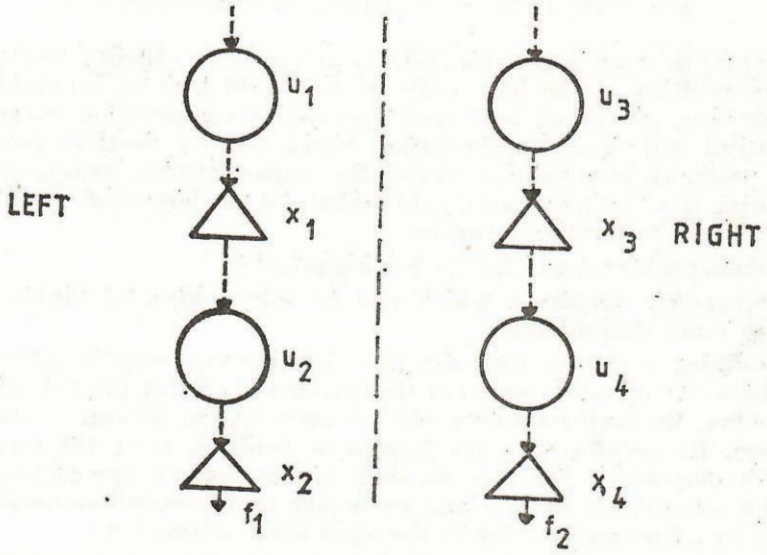


Fig. 1. *Dynamic model for production management, planning and scheduling**

and proceeding to its decomposition and co-ordination by the method of vertical decomposition, we obtain:

The dynamic model of the system:

— the dynamics:

$$x_1(k+1) = x_1(k) + u_1(k) - u_2(k)$$
$$x_2(k+1) = x_2(k) + u_2(k) - f_1(k)$$
$$x_3(k+1) = x_3(k) + u_3(k) - u_4(k)$$
$$x_4(k+1) = x_4(k) + u_4(k) - f_2(k)$$

— capacity constraints

$$u_1(k) \leqslant d_{M1}(k)$$
$$u_2(k) + u_4(k) \leqslant d_{M3}(k)$$
$$u_3(k) \leqslant d_{M2}(k)$$

* The modelled system is the contribution of P. Fomin.

— continuity constraints in the technological flow (succession constraints)

$$x_1(k) \geqslant u_2(k)$$

$$x_2(k) \geqslant f_1(k)$$

$$x_3(k) \geqslant u_4(k)$$

$$x_4(k) \geqslant f_2(k)$$

— performance function

$$\max_{x_i, u_i} \left\{ \sum_{k=0}^{N-1} [c_1 x_i(k) + c_2 u_i(k)] \right\}$$

— the initial state

$$x_1(0) = 0$$

$$x_2(0) = 0$$

$$x_3(0) = 0$$

$$x_4(0) = 0$$

— capacity of equipment, supply and delivery

d and f \ k	0	1	2	3	4
$M1$		400	200		
$M2$	400		200		
$M3$		400	400	400	
f_1					600
f_2					600

If we analyse the data of the modelled system, we notice that it has two finished products at the output (f_1 and f_2) and the solution of the system is unique (it has only one admissible solution and this is the optimal solution as well), that is the polyhedron of the solutions is pointlike.

Decomposing the system by the method of vertical decomposition, presented above, we obtain two main subproblems equal in length (control elements) and without links (nodes).

Thus the system decomposed by means of vertical decomposition, presents the main subproblem "left" and the main subproblem "right".

The main subproblem "left" is formalized in the following way:

— the dynamics

$$x_1(k+1) = x_1(k) + u_1(k) - u_2(k)$$

$$x_2(k+1) = x_2(k) + u_2(k) - f_1(k)$$

— capacity constraints

$$u_1(k) \leqslant d_{M1}(k)$$

$$u_2(k) \leqslant d_{M3}(k)$$

— continuity constraints in the technological flow (succession constraints)

$$x_1(k) \geqslant u_2(k)$$

$$x_2(k) \geqslant f_1(k)$$

— performance function

$$\max_{x_i, u_i} \left\{ \sum_{k=0}^{N-1} [c_1 x_i(k) + c_2 u_i(k)] \right\}$$

— the initial state

$$x_1(0) = 0$$

$$x_2(0) = 0$$

— capacity of equipment, supply and delivery

k / d and f	0	1	2	3	4
$M1$		400	200		
$M3$		400	400	400	
f_1					600

The main subproblem "right" is the following:
— the dynamics

$$x_3(k+1) = x_3(k) + u_3(k) - u_4(k)$$

$$x_4(k+1) = x_4(k) + u_4(k) - f_2(k)$$

— capacity constraints

$$u_3(k) \leqslant d_{M2}(k)$$

$$u_4(k) \leqslant d_{M3}(k) - u_2(k)$$

— continuity constraints in the technological flow (succession constraints)

$$x_3(k) \geqslant u_4(k)$$

$$x_4(k) \geqslant f_2(k)$$

— initial state

$$x_3(0) = 0$$

$$x_4(0) = 0$$

— capacity of equipment, supply and delivery

d and f \ k	0	1	2	3	4
$M2$	400		200		
$M3$		$400 - u_2(1)$	$400 - u_2(2)$	$400 - u_2(3)$	
f_2					600

— performance function

$$\max_{x_i\ u_i}\left\{\sum_{k=0}^{N-1} c_1 x_i(k) + c_2 u_i(k)]\right\}$$

OBTAINED RESULTS

We consider that the system presented in Fig. 1 is a large-scale system and we have solved the problem both globally and by decomposition, using the method of vertical decomposition, for the two main subproblems.

a) The performance function used:

$$\max_{x_i\ u_i}\left\{\sum_{k=0}^{N-1} [c_2 u_i(k)]\right\}$$

$$c_{21} = 10 \quad c_{23} = 5$$

$$c_{22} = 15 \quad c_{24} = 35$$

The global solving of the system has led to the following results:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 39,000 lei.

The decomposed solving of the system by the method of vertical decomposition has led to the following results:

— for the main subproblem "left"

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 15,000 lei.

— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has the value of 24,000 lei.

Thus, the sum of partial optima for the two subproblems (15,000 + 24,000) equals the global optimum (39,000).

Consequently, the method of vertical decomposition leads to an optimal solving of large-size systems (the sum of partial optima equals the global optimum). At the same time, the evolution of states and controls for the two subproblems is identical to the one in the problem solved globally.

b) The problem was solved also for another performance function:

$$\max_{x_i, u_i} \left\{ \sum_{k=0}^{N-1} [c_1 x_i(k) + c_2 u_i(k)] \right\}$$

$$c_{11} = 35 \quad c_{21} = 10$$

$$c_{12} = 50 \quad c_{22} = 15$$

$$c_{13} = 40 \quad c_{23} = 5$$

$$c_{14} = 75 \quad c_{24} = 35$$

The global solving of the system has led to the following results:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 239,000 lei.

The solving by decomposition, using the method of vertical decomposition, has led to the following results:

— for the main subproblem "left"

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 86,000 lei.

— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_3	400	0	200	0	0	
u_1	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 153,000 lei.

The sum of partial optima for the two main subproblems (86,000 + 153,000) equals the global optimum of 239,000 lei.

c) The system was solved also for a third performance function:

$$\max_{x_i}\left\{\sum_{k=0}^{N-1}[c_1 x_i(k)]\right\}$$

The global solving of the system has led to the following results:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has the value of 200,000 lei.

The solving by decomposition, using the method of vertical decomposition, has led to the following results:

— for the main subproblem "left"

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 71,000 lei.

— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 129,000 lei.

The sum of the partial optima for the two main subproblems, (71,000 + 129,000), equals the global optimum of 200,000 lei.

Obtained results emphasize that the used performance functions — a), b) and c) — are similar, and that the evolution of states and controls (for the subproblems, as compared to the overall problem) is identical. A single difference is to be noted, namely the value of the performance function when penalty coefficients are used, i.e. — a) only for controls, b) for states and controls, c) only for states.

d) Using the performance function:

$$\max_{u_i}\left\{\sum_{k=0}^{N-1}[c_2 u_i(k)]\right\}$$

$$c_{21} = 10 \quad c_{23} = 5$$

$$c_{22} = 35 \quad c_{24} = 35$$

for the global system, following results are obtained:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 51,000 lei.

Solving by vertical decomposition led to the following results:

— for the main subproblem "left"

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 27,000 lei.

— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_2	400	0	200	0	0	
u_4	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 24,000 lei.

The sum of the partial optima for the two subproblems (27,000 + 24,000) is equal to the global optimum (51,000 lei).

Hence, also in the case when the penalty costs in the performance function are equal for distinct operations performed on the same equipment, but in the frame of different subproblems, the sum of the partial optima pertaining to the subproblems is equal to the global optimum, providing thus a maximal economic efficiency. In this case too, evolution of states and orders in the subproblems is identical to that global system.

Also for the performance functions:

$$\max_{x_i,u_i}\left\{\sum_{k=0}^{N-1}[c_1x_i(k) + c_2u_i(k)]\right\}$$

and respectively

$$\max_{x_i}\left\{\sum_{k=0}^{N-1}[c_1x_i(k)]\right\}$$

optimal results are obtained for the two subproblems, results which summed up are equal to the global optimum.

e) Using the performance function:

$$\min_{u_i}\left\{\sum_{k=0}^{N-1}[c_2u_i(k)]\right\}$$

for the global system, following results are obtained:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 51,000 lei.
The decomposed solving of the system led to the following results:
— for the main subproblem "left"

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 27,000 lei.
— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 24,000 lei.
The sum of the partial optima for the two subproblems (27,000 + 24,000) is equal to the global optimum (51,000 lei).

Thus, also in this case an optimal solution (the sum of the values of the performance functions pertaining to the subproblems is equal to the global optimum) is obtained by using the penalty coefficients of d).

It is obvious that also for the performance functions

$$\min_{x_i,u_i}\left\{\sum_{k=0}^{N-1}[c_1x_i(k) + c_2u_i(k)]\right\}$$

and respectively

$$\min_{x_i}\left\{\sum_{k=0}^{N-1}[c_1x_i(k)]\right\}$$

identical results are obtained, the sum of subproblem optima being equal to the global optimum.

f) Using the performance function

$$\min_{u_i}\left\{\sum_{k=0}^{N-1}[c_2u_i(k)]\right\}$$

and penalty coefficients of a), following results are obtained for the global system:

u and x \ k	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 39,000 lei.
Decomposed solving of the system led to following results:
— for the main subproblem "left"

u and x	1	2	3	4	5	6
u_1	0	400	200	0	0	
u_2	0	0	400	200	0	
x_1	0	0	400	200	0	0
x_2	0	0	0	400	600	0

The performance function has a value of 15,000 lei.
— for the main subproblem "right"

u and x \ k	1	2	3	4	5	6
u_3	400	0	200	0	0	
u_4	0	400	0	200	0	
x_3	0	400	0	200	0	0
x_4	0	0	400	400	600	0

The performance function has a value of 24,000 lei.

In this use too, the sum of subproblem optima (15,000 + 24,000) is equal to the global optimum (39,000).

Also for the performance functions

$$\min_{x_i, u_i} \left\{ \sum_{k=0}^{N-1} [c_1 x_i(k) + c_2 u_i(k)] \right\}$$

and respectively

$$\min_{x_i} \left\{ \sum_{k=0}^{N-1} [c_1 x_i(k)] \right\}$$

optimal results are obtained for the subproblems "left" and respectively "right", results which summed-up are equal to the global optimum.

5. CONCLUSIONS

It follows from the above-mentioned results that, having to solve a large-scale problem, on existing equipment and by means of existing optimization packages, the sole means available is decomposition into subproblems.

The proposed method — namely vertical decomposition — allows both to reduce the size of the large problem, tallying it to the capabilities of computers and optimization packages, and to co-ordinate the subproblems, so that the sum of subproblem optima should be equal to the optimum of the large problem.

The method is illustrated by a practical exemple which shows how the decomposition of the system, respectively co-ordination of the subproblems, is achieved.

Finally, it may be emphasized that:

— large-scale problem solving by vertical decomposition takes into account both constraints and performance function to be extremized;

— vertical decomposition can be applied to problems having fixed requirements as regards outputs;

— if outputs are free, depending upon the wished extremum (minimum or maximum), the link between the physical process (material consumption, equipment utilization) and the function to be optimized is easier to grasp and will reflect itself in the results obtained by global system solving, respectively by decomposition into subproblems.

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UN PROCÉDÉ OPTIMAL POUR LA RÉSOLUTION DES PROBLÈMES DE GRANDES DIMENSIONS: LA DÉCOMPOSITION VERTICALE

On présente un procédé de décomposition des grands systèmes en produits, ensemblies, sous-ensembles, semi-produits et parties complexes utilisant la méthode de la décomposition verticale.

La décomposition des problèmes de grandes dimensions en problèmes relativement petits rend possible leur résolution utilisant les configurations de calculs courants et les paquets d'optimisation existents. La méthode de la décomposition verticale permet la coordination optimale des grands systèmes tant par l'intermédiaire des restrictions que des fonctions de performance

A la fin, on présente les résultats obtenus pour un système dont le modèle dynamique pour la planification et la programmation de la production a été décomposé et coordiné de façon optimale.

ОПТИМАЛЬНЫЙ СПОСОБ РЕШЕНИЯ ЗАДАЧ КРУПНОГО РАЗМЕРА; ВЕРТИКАЛЬНОЕ РАЗЛОЖЕНИЕ

Работа приводит способ разложения крупных систем в изделия, ансамблы, подансамблы, полуизделия и комплексные части, при помощи метода вертикального разложения.

Разложение задач крупных размеров в относительно небольшие задачи создаёт возможность их разрешения, используя обычные расчётные системы и существующие программные пакеты для оптимизации. Метод вертикального разложения позволяет произвести оптимальную координацию крупных систем посредством, так ограничений, как и оптимизационных функций.

В конце статье приводятся результаты, полученные для системы, чЪей динамическая модель планирования и программирования производства была оптимально разложена и скоординирована.

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