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On an Entropic Solution of Co-operative Games

An entropic solution of a co-operative game is given, starting from minimizing the sum of the gain non-determinations by the decision-makers, participating in the co-operative game. The entropic solution is approximated by solving a non-linear programme.

The co-operative games occupy a special place in the conflictual model theory, by their very content; determining the solution of a co-operative game is, in the last analysis, a distribution problem, whose solution must be optimal for all decision-makers. Several solutions (Shapley value, negotiation set, stable sets a.s.o.) are remarked in the theory of games, as the notion of optimum is linked to the use of an optimality criterion; each of the above mentioned solutions presents obvious drawbacks, connected either to the lack of generality of the obtained solution, or to the very conceptual framework of approaching the problem.

We present further down a concept of solving co-operative games,which leads finally to a non-linear programming problem.

Let us suppose that coalition C is formed from n decision-makers; magnitudes V_i and a_i are attached to each decision-maker i, $i = \overline{1, n}$ and their significances are as follows:

V_i = maximin utility, i.e. the certain gain value

a_i = coalition utility.

There are following possibilities, in this situation, if we denote by X_i the total gain of decision-maker i:

$$X_i \leqslant a_i, \text{ if the total gain } X \leqslant \sum_{i=1}^{n} ai, \; i = \overline{1, n}$$

$$X_i > a_i, \text{ if the total gain } X > \sum_{i=1}^{n} ai, \; i = \overline{1, n}.$$

We consider the non-determination function — Watanabe's inaccuracy — introduced by Tövissi in [3]:

$$F(x,\ \alpha) = -\ x^{\alpha}\ 1n\ x - (1 - x)^{\alpha} - 1n(1 - x) + \frac{1 - \alpha}{\alpha}[x^{\alpha} +$$

$$+ (1 - x)^{\alpha} - 1]\alpha \in (0, 1).$$

Definition

The solution of the co-operative game is vector $(X_1, X_2, ..., X_n)$ which minimizes the total non-determination

$$\widetilde{F}(x_1, x_2, ..., x_n, \alpha) = \sum_{i=1}^{n} F(x_i, \alpha), \text{ where } x_1, x_2, ..., x_n$$

are built according to the procedure indicated in [2]:

$$x_i = \frac{X_i - V_i}{a_i - V_i},\ \ i = \overline{1, n}\cdot$$

Remark 1: The significance of the introduced solution is connected to the fact that, as the decision-makers make up a coalition with a certain organizating degree W, it is rational to distribute to each decision-maker a gain such that the deviations from the distributed gains and the theoretical ones give a minimal total inaccuracy, a fact which can be achieved with the help of function $\widetilde{F}$.

On the other hand, as we shall see, function $\widetilde{F}$ can be well approximated through function

$$F^*(x_1, x_2, ..., x_n, \alpha) = \frac{1 + 3\ n\alpha - 3\alpha}{\alpha} \sum_{i=1}^{n} H_i + \frac{n}{\alpha}$$

where H_i is the gain non-determination for the gambler i, $i = \overline{1, n}$.

Proposition

a) The approximate solution of the co-operative game, in the case $X \leqslant \sum_{i=1}^{n} a_i$, is given by

$$X_i = x_i^0(a_i - V_i) + V_i,\quad i = \overline{1, n}$$

where $(x_1^0, x_2^0, ..., x_n^0)$ is the solution of the problem.

$$min\ F^*(x_1, x_2, ..., x_n)$$

$$\sum_{i=1}^{n} C_i x_i = Y$$

$$x_i \geqslant 0,\ \ i = 1, n \qquad (1)$$

b) The approximate solution of the co-operative game, in the case $X > \sum_{i=1}^{n} a_i$, is given by $X_i = \frac{a_i - V_i}{x_i^0} + V_i$, $i = \overline{1,n}$ where $(x_1^0, x_2^0, ..., x_n^0)$ is the solution of the problem

$$min\, F^*(x_1, x_2, ..., x_n)$$

$$\sum_{i=1}^{n} \frac{C_i}{x_i} = Y$$

$$x_i \geqslant 0,\ i = \overline{1, n}. \tag{2}$$

We denoted $C_i = a_i - V_i$, $Y = X - \sum_{i=1}^{n} V_i$.

Proof. Case a) corresponds to the distribution in Fig. 1.

Denoting by $x_i = \frac{X_i - V_i}{a_i - V_i} \rightarrow x_i(a_i - V_i) = X_i - V_i$ as $a_i - V_i = C_i \Rightarrow \sum_{i=1}^{n} C_i x_i = X - \sum_{i=1}^{n} V_i$ and as $X - \sum_{i=1}^{n} V_i = Y \Rightarrow \sum_{i=1}^{n} C_i x_i = Y$, the constraints of problem (1) result therefrom.

Case b) corresponds to the distribution in Fig. 2.

Denoting $x_i = \frac{a_i - V_i}{X_i - V_i} \Rightarrow X_i - V_i = \frac{a_i - V_i}{x_i} \Rightarrow Y = \sum_{i=1}^{n} \frac{C_i}{x_i}$ the constraints of problem (2) result therefrom.

Remarks: 2) Case a) corresponds to the situation when the decision-makers of the coalition obtain smaller gains than the propesed ones, and case b) corresponds to the situation when the decision-makers obtain higher gains than the proposed ones. These two situations are unique and they result by comparing magnitudes X and $\sum_{i=1}^{n} a_i$.

3) Both cases lead to non-linear programming problems, which is a large gain in comparison with the above mentioned difficulties, connected to the solution of a co-operative game.

Let us prove the following proposition, before proving that F^* is a good approximation of $\widetilde{F}$:

Proposition

$$F(x,\ \alpha) = \frac{1-\alpha}{\alpha} e^{H(1-\alpha)} - \frac{\partial}{\partial \alpha}\ (e^{H(1-\alpha)}) - \frac{1-\alpha}{\alpha},$$

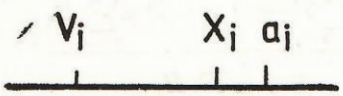

Fig. 1

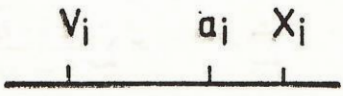

Fig. 2

H being the entropy, in Rényi's sense.

$$H(x,\alpha)=\begin{cases}\dfrac{1}{1-\alpha}\ln[x^{\alpha}+(1-x)^{\alpha}], & \alpha\neq 1\\ -x\ln x-(1-x)\ln(1-x), & \alpha=1.\end{cases}$$

Proof.

$$H'_{\alpha}=\frac{1}{(1-\alpha)^2}\ln[x^{\alpha}+(1-x)^{\alpha}]+\frac{1}{(1-\alpha)}\left[\frac{x^{\alpha}\ln x+(1-x)^{\alpha}\ln(1-x)}{x^{\alpha}+(1-x)^{\alpha}}\right]=$$

$$=\frac{1}{1-\alpha}\left[\frac{x^{\alpha}\ln x(1-x)^{\alpha}\ln(1-x)}{x^{\alpha}+(1-x)^{\alpha}}\right]+H$$

$$F(x,\alpha)=\frac{1-\alpha}{\alpha}e^{H(1-\alpha)}+He^{H(1-\alpha)}-H'_{\alpha}(1-\alpha)\,e^{H(1-\alpha)}-\frac{1-\alpha}{\alpha}=$$

$$=\frac{1-\alpha}{\alpha}e^{H(1-\alpha)}-\frac{\partial}{\partial\alpha}(e^{H(1-\alpha)})-\frac{1-\alpha}{\alpha}$$

quod erat demonstrandum.

We shall denote now by H_i the gain entropy of decision-maker i(it is obvious that H_i shall be Rényi's entropy),

$$\widetilde{F}(x_1,\ x_2,\ \ldots,\ x_n\alpha)=\sum_{i=1}^{n}F(x_i,\alpha)=\frac{1-\alpha}{\alpha}\sum_{i=1}^{n}e^{H_i(1-\alpha)}$$

$$-\frac{\partial}{\partial\alpha}\left(\sum_{i=1}^{n}e^{H_i(1-\alpha)}\right)-\sum_{i=1}^{n}\frac{1-\alpha}{\alpha}\ \text{(after an immediate calculation).}$$

Remark 4. We shall consider further down $\alpha<\frac{1}{2}$, from the considerations exposed in [2], and we shall neglect the products between overunitary powers of α and H_i, as well as the overunitary powers of H_i, $i=\overline{1,n}$. We shall have then:

$$\sum_{i=1}^{n}e^{H_i(1-\alpha)}\sum_{i=1}^{n}\frac{e^{H_i}}{e^{\alpha H_i}}=\sum_{i=1}^{n}\frac{1+\dfrac{H_i}{1!}+\dfrac{H_i^2}{2!}+\ldots}{1+\dfrac{\alpha H_i}{1!}+\dfrac{\alpha^2H_i^2}{2!}+\ldots}\qquad(3)$$

According to remark 4, relation (3) can be approximated through:

$$\sum_{i=1}^{n}\frac{1+H_i}{1+\alpha H_i}$$

$$\sum_{i=1}^{n}e^{H_i(1-\alpha)}\approx\sum_{i=1}^{n}\frac{1+H_i}{1+\alpha H_i}\approx\frac{n+\sum_{i=1}^{n}H_i+(n-1)\,\alpha\sum_{i=1}^{n}H_i}{1+\alpha\sum_{i=1}^{n}H_i}.\qquad(4)$$

We take into account (4) and remark 4 for determining

$$\frac{\partial}{\partial \alpha}\left(\sum_{i=1}^{n} e^{H_i(1-\alpha)}\right).$$

We obtain finally

$$\frac{\partial}{\partial \alpha}\left(\sum_{i=1}^{n} e^{H_i(1-\alpha)}\right) \approx \frac{\partial}{\partial \alpha} \frac{n+\sum_{i=1}^{n} H_i + (n-1)\alpha \sum_{i=1}^{n} H_i}{1+\alpha \sum_{i=1}^{n} H_i}$$

$$\frac{(1-\alpha)\left(\sum_{i=1}^{n} H_i\right)^1 \alpha - \sum_{i=1}^{n} H_i - \left(\sum_{i=1}^{n} H_i\right)^2}{1+2\alpha \sum_{i=1}^{n} H_i}.$$

$F^*(x_1, x_2, ..., x_n^{\alpha})$ can be, at last, determined, by neglecting all terms wherein there appear products between the powers of α and the subunitary magnitudes $\sum_{i=1}^{n} H_i, \left(\sum_{i=1}^{n} H_i\right)'_{\alpha}$ (excepting those products which contain, besides the mentioned factors, also the powers of n). We obtain finally, after an immediate calculation

$$F^*(x_1, x_2, ..., x_n, \alpha) = \frac{(1+3n\alpha-3\alpha)\sum_{i=1}^{n} H_i}{\alpha} - \frac{n}{\alpha}.$$

We can draw immediately the conclusion from the form of F^*, that minimizing F^* is equivalent to minimizing the sum of the non-determinations of the gain of the n gamblers; in other words expressed, $(x_1^0, x_2^0, .., x_n^0)$ is the solution of the given problem in the context of the constraints of cases a) or b), and so we have:

$$\sum_{i=1}^{n} H(x_i^0, \alpha) = min \sum_{i=1}^{n} H(x_i, \alpha).$$

Taking into account that the organization degree in Watanabe's sense is $W = \sum_{i=1}^{n} H_i - \mathcal{H}$, $\mathcal{H}$ being the non-determination of the aggregate formed by the n decision-makers, in the case when we know the organization degree of the system formed by the n gambler, and the optimal distribution determining problem becomes:

Case a)

$$\begin{cases} min\,[\mathcal{H}(x_1, x_2, ..., x_n, \alpha) + W] \\ \sum_{i=1}^{n} C_i x_i = Y \\ x_i \geqslant 0,\ i = \overline{1, n} \end{cases}$$

Case b)

$$\begin{cases} min\,[\mathcal{H}(x_1, x_2, ..., x_n, \alpha) + W] \\ \sum_{i=1}^{n} \frac{C_i}{x_i} = Y \\ x_i \geqslant 0,\ i = \overline{1, n}. \end{cases}$$

We are also lead to two non-linear programming problems, whose numerical solving can be made with the help of known methods, which can be algorithmated and programmed in an electronic computer.

REFERENCES

1. GUIAŞU, S., MALIŢA, M.: *Triade* (Triads). Ed. ştiinţifică, Bucureşti, 1973.
2. OWEN, G.: *Teoria jocurilor* (Theory of Games). Ed. tehnică, Bucureşti, 1974.
3. THEILER G., TÖVISSI, L.: *Măsuri entropice neprobabilistice, măsuri ale impreciziei*. (Entropic Non-probabilistic Measures, Measures of Inaccuracy). Rev. Statistică, no. 5/1978.

SUR UNE SOLUTION ENTROPIQUE DES JEUX COOPERATIFS

On donne une solution entropique d'un jeu coopératif prenant pour point de départ la minimisation de la somme des non-déterminations des gains. La solution entropique est approximée par la résolution d'un programme non-linéaire.

ОБ ОДНОМ ЭНТРОПИЧЕСКОМ РЕШЕНИИ КООПЕРАТИВНЫХ ИГР

В статье приводится энтропическое решение кооперативной игры, исходя из минимизации суммы неопределённостей выигрыша, энтропическое решшение апроксимируется при помощи нелинейной программы.

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