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On Some Random Functions Almost Periodic in Bohr Sense

This paper deals with random functions f with vectorial values ρ — almost periodic in Bohr sense, and functions f, weakly almost periodic in probability in Bohr sense, and shows that the property of almost periodicity in Bohr sense, is implied by the existence of a sequence. $(\delta_k)_{k\in N^*}$, $\delta_k > 0$, $\delta_k \searrow 0$ *such that the sequence of variables* $g(\omega) = f(n\,\delta_k, \omega)$ $n \in Z$ *be ρ — almost periodic or weakly almost periodic for any* $k \in N^*$.

1. Let $\{\Omega, K, P\}$ be a probabilistic field and X the set of all n-dimensional random vectors, i.e. functions $x : \Omega \to C^n$. We introduce, as in [9] [11], on space X the semymmetrical ρ, where

$$\rho(x, y) = \inf_{\alpha>0} \{\alpha + P(\{\omega : \| x(\omega) - y(\omega \| \geqslant \alpha\})\}$$

as against to which space X is a Fréchet space. Papers [6], [7] introduced the notions of random function ρ-almost periodic in Bohr sense, as well as a sequence of random variables ρ-almost periodic, through:

Definition 1.1. Random function $f : R \to X$, ρ — continuous is ρ-almost periodic in Bohr sense, if for any $\varepsilon > 0$ there is $l(\varepsilon) > 0$ such that in any interval $I \subset R$ of length $l(\varepsilon)$ there is a number τ with the property that $\rho(f(t + \xi\tau), f(t)) < \varepsilon$ whichever $t \in R$ would be.

Remark. Number τ is called — $(\rho - \varepsilon)$ almost periodic.

Definition 1.2. Sequence $(f_n)_{n\in Z} \subset X$ is ρ-almost periodic if for any $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that any set of $N(\varepsilon)$ consecutive integers contains an integer p with the property that: $\rho(f_{n+p}, f_n) < \varepsilon$ whichever $n \in Z$ would be.

Remark. We denoted by Z the set of integer numbers. Dealing as in paper [4], we give the result:

Proposition 1.1. Let $f : R \to X$, be the random function, ρ-uniformly continuous. If there is a sequence $(\delta_k)_{k\in N}$, $\delta_k > 0$, monotonously decreasing converging toward zero, such that sequence $g_n(\omega) = f(n\delta_k, \omega)$, $n \in Z$ is δ-almost periodic for any $k \in N$, then random function f is ρ-almost periodic in Bohr sense.

Proof. Let be $\varepsilon > 0$. There is $\delta(\varepsilon) > 0$ such that $\rho(f(t'), f(t'')) < \frac{\varepsilon}{8}$ if $|t' - t''| < \delta(\varepsilon)$. We select $k(\varepsilon) \in N$ such that, for any $k \geqslant k(\varepsilon)$ we have $\delta_k < \delta(\varepsilon)$. We state $k_0 \in N$, $k_0 \geqslant k(\varepsilon)$ and consider the sequence of random variables $(g_n(\omega))_{n\in Z}$ $g_n(\omega) = f(n, \delta_{k0}, w)$ wehich, by hypothesis, is ρ-almsot periodic. There is, hence, natural number $N(\varepsilon)$, such that any set of $N(\varepsilon)$ consecutive integer numbers contains an integer number p with the property that $\rho(g_{n+p}, g_n) < \frac{\varepsilon}{8}$ whichever $n \in Z$ would be. Let p be such an integer number, fixed, $\tau = p\delta_{k0}$ and $t \in R$. There is $n \in Z$ such that $n\delta_{k0} \leqslant t < (n+1)\delta_{k0}$. Then, $\left\{\omega : \|f(t+\tau, \omega) - f(t, \omega)\| > \frac{\varepsilon}{2}\right\} \subset \left\{\omega : \|f(t+\tau, \omega) - f(n\delta_{k0}+\tau, \omega)\| > \frac{\varepsilon}{6}\right\} U \left\{\omega : \|f(n\delta_{k0}, \omega) - f(n\delta_{k0}+\tau, \omega)\| > \frac{\varepsilon}{6}\right\} U \left\{\omega : \|f(n\delta_{k0}, \omega) - f(t, \omega)\| > \frac{\varepsilon}{6}\right\}$. It results : $P(\{\omega : \|f(t+\tau, \omega) - f(t, \omega)\| > \frac{\varepsilon}{2}\}) \leqslant P(\{\omega : \|f(t+\tau, \omega) - f(n\delta_{k0}+\tau, \omega)\| > \frac{\varepsilon}{6}\}) + P(\{\omega : \|f(n\delta_{k0}+\tau, \omega) - f(n\delta_{k0}, \omega)\| > \frac{\varepsilon}{6}\}) + P\left(\left\{\omega : \|f(n\delta_{k0}, \omega) - f(t, \omega)\| > \frac{\varepsilon}{6}\right\}\right) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$.

It results from [7] that $\rho(f(t+\tau), f(t)) < \varepsilon$, hence $\tau = p\delta_{k0}$ is a (ρ — — ε)-almost periode.

2. Let X be a Banach space, X' its dual, and $\mathfrak{L}(\Omega, X, K, P)$ the set of functions defined on Ω weakly measurable. Let us fix $M \subset X'$, M non-void. Papers [3], [5] introduced the notions of random function weakly almost periodic in probability in Bohr sense with respect to M, and the notion of sequence of functions weakly almost periodic in probability with respect to M, through the following definitions:

Definition 2.1. Function $f : R \to \mathfrak{L}(\Omega\ X, K, P)$ M — continuous in probability on R, is weakly almost periodic in probability in Bohr sense with respect to M, if for any $\varepsilon > 0$, $\eta > 0$ there is $l(\varepsilon, \eta) > 0$, such that in any interval $I \subset R$ of length $l(\varepsilon, \eta)$ there is a point τ with the property: $\sup_{x' \in M} P(\{\omega : |x'(f(t, \omega)) - x'(f(t+\tau, \omega))| \geqslant \varepsilon\}) \leqslant \eta$ for any $t \in R$.

Point τ from definition 2.1 is called, by analogy with the classical case, (ε, η) weakly almost periodic in probability with respect to M.

Definition 2.2. The sequence of variables $(f_n)_{n\in Z} \subset \mathfrak{L}(\Omega, X, K, P)$ is weakly almost periodic in probability with respect to M, if for any $\varepsilon > 0$, $\eta > 0$ there is the natural number $N(\varepsilon, \eta)$ such that in any set of $N(\varepsilon, \eta)$ consecutive integer numbers there is an integer number p having the property that $\sup_{x' \in M} p(\{\omega : |x'(f^{(\omega)}_{n+p}) - x'(f^{(\omega)}_n)| \geqslant \varepsilon\}) \leqslant \eta$ whichever $n \in Z$ would be.

We have the result:

Proposition 2.1. Let $R \to L(\Omega, X, K, P)$ be a random function such that the family $\{x'\circ, f\}_{x'\in M}$ is equicontinuous in probability uniformly with respect to $t \in R$. If there is a sequence of numbers $(\delta_k)_{k\in N}$, $\delta_k > 0$ monotonously decreasing, converging toward zero with the property that sequence $g_n(\omega) = f(n\delta_k, \omega)$ is weakly almost periodic in probability with respect to M for any $k \in N$, then function f is weakly almost periodic in probability in Bohr sense with regard to M.

Proof. Let $\varepsilon > 0$, $\eta > 0$. There is $\delta(\varepsilon, \eta) > 0$ such that

$$\sup_{x' \in M} P\{\omega : |x'(f(t', \omega)) - x'(f(t'', \omega))| \geqslant \frac{\varepsilon}{3}\} \leqslant \eta/3$$

for any $t', t'' \in R$, with $|t' - t''| < \delta(\varepsilon, \eta)$. We select a natural number $k(\varepsilon, \eta)$ such that $\delta_k \leqslant \delta(\varepsilon, \eta)$ for any $k \geqslant k(\varepsilon, \eta)$. Let $k_0 \geqslant k(\varepsilon, \eta)$ be fixed. We consider sequence $g_n(\omega) = f(n\delta_{k0}, \omega)$ $n \in Z$. By hypothesis, sequence $(g_n(\omega))_{n\in Z}$ is weakly almost periodic in probability with respect to M. Hence, there is natural number $N(\varepsilon, \eta)$ with the property that any set of $N(\varepsilon, \eta)$ consecutive integers contains an integer number p, such that

$$\sup_{x' \in M} P\left(\left\{\omega : |x'(g^{(\omega)}_{n+p}) - x'(g^{(\omega)}_n)| \geqslant \frac{\varepsilon}{3}\right\}\right) \leqslant \frac{\eta}{3}$$

for any $n \in Z$. Let p be such an integer number and $\tau = p\delta_{k0}$. Let also $t \in R$ be arbitrary. There is $n \in Z$ such that $n\delta_{k0} \leqslant t < (n+1)\delta_{k0}$. Then,

$$P(\{\omega : |x'(f(t+\tau, \omega)) - x'(f(t, \omega))| \geqslant \varepsilon\}) \leqslant P(\{\omega : |x'(f(t+\tau, \omega) -$$

$$- x'(f(n\delta_{k0} + \tau, \omega))| \geqslant \frac{\varepsilon}{3}\}) + P\left(\left\{\omega : |x'(f(n\delta_{k0} + \tau, \omega)) - x'(f(n\delta_{k0}, \omega))| \geqslant\right.\right.$$

$$\left.\left.\geqslant \frac{\varepsilon}{3}\right\}\right) + P\left(\left\{\omega : |x'(f(n\delta_{k0}, \omega)) - x'(f(t, \omega))| \geqslant \frac{\varepsilon}{3}\right\}\right) \leqslant \sup_{x' \in M} P\left(\left\{\omega : |x'(f(t+\tau, \omega)) -\right.\right.$$

$$\left.\left.- x'(f(n\delta_{k0} + \tau, \omega))| \geqslant \frac{\varepsilon}{3}\right\}\right) + \sup_{x' \in M} P\left(\left\{\omega : |x'(f(n\delta_{k0} + \tau, \omega)) - x'(f(n\delta_{k0}, \omega))| \geqslant\right.\right.$$

$$\left.\left.\geqslant \frac{\varepsilon}{3}\right\}\right) + \sup_{x' \in M} P\left(\left\{\omega : |x'(f(n\delta_{k0}, \omega)) - x'(f(t, \omega))| \geqslant \frac{\varepsilon}{3}\right\}\right) \leqslant \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$

Hence: $\sup_{x' \in M} P(\{\omega : |x'(f(t+\tau, \omega)) - x'(f(t, \omega))| \geqslant \varepsilon\}) \leqslant \eta$ for any $t \in R$ and $\tau = p\delta_{k0}$ is a $-(\varepsilon, \eta)$ weakly near period with respect to M for function $f: R \to L(\Omega, X, K, P)$.

REFERENCES

1. BOHR H.: *On the definition of almost periodicity.* Journal d'Analyse Mathématique, 1 11—27, 1951.
2. CENUŞĂ GH., SACUIU I.: *On some stochastic approximation for random functions.* Rediconti di Matematica, Vol. 12, seria VI, 1979, pp. 143—155.
3. CENUŞĂ GH.: *Funcţii slab aproape periodice în probabilitate.* (Functions weakly almost periodic in probability). To appear next. Revista Studii şi cercetări matematice, nr. 4, 1981.
4. CENUŞĂ GH. SACUIU I.: *Sequences of random variables almost periodic in probability.* Economic computation and economic cybernetics studies and research. Nr. 2, 1979.
5. CENUŞĂ GH.: *Funcţii simptotice slab aproape periodice în probabilitate* (Symptotic functions weakly almost periodic in probability). To appear next. Revista Studii şi cercetări matematice, nr. 4, 1981.
6. CENUŞĂ GH.: *Şiruri de variabile aleatoare* ρ *— aproape periodice.* (Sequences of almost periodic random variables). To apear next. Studii şi cercetări de calcul economiec şi cibernetică economică, nr. 1, 1980, Bucureşti.
7. CENUŞĂ GH.: *Random vector functions almost periodic.* To appear next. Buletinul Societăţii de matematică din R.S.R. nr. 2 1980.
8. CORDUNEANU C.: *Funcţii aproape periodice* (Almost periodic functions). Editura Academiei R.S.R. 1961.
9. MOROZAN T.: *Stabilitatea sistemelor cu parametri aleatori* (Stability of random parameter systems). Editura Academiei R.S.R., 1969.
10. ONICESCU O., ISTRATESCU V.: *Approximation theorem for random functions.* Rediconti di matematica, 8, serie VI, pp. 65—81.
11. DUNFORD M., SWARTZ D.: *Linear operators.* Part I. New York, 1958.

CONCERNANT CERTAINES FONCTIONS ALEATOIRES PRESQUE PERIODIQUES EN SENSE BOHR

Dans cet ouvrage on considère des fonctions aléatoires f à valeur vectorielle ρ — presquse périodiques en sens Bohr et des fonctions f faiblement presque périodiques en probabilité en sens Bohr, mettant en évidence que la propriété de presque périodique en sens Bohr est impliquée par l'existence d'une suite $(\delta_k)_{\widetilde{k}\in N^*}$ $\delta_k>0$, $\delta_k\searrow 0$ de telle manière que la suite des variables $g_n(w) = f(n\ \delta_k,\ w) n \in Z$ soit ρ — presque périodique ou faiblement presque périodique pour tout $K\in\ N^*$.

К НЕКОТОРЫМ АЛЕАТОРНЫМ ФУНКЦИЯМ, ПОЛУПЕРИОДИЧНЫМ В СМЫСЛЕ БОХР

В этой работе считают алеаторными функциями функции с вектриальными значениями ρ, полупериодичные в смысле Бохр, и функции δ, слаба полупериодичные по вероятности в смысле Бохр. Свойство полупериодичности в смысле Бохр предполагает существование ряда $(\delta_k)_{k\in N^*}$, $\delta_k > 0$, $\delta_k\searrow 0$ так, чтобы ряд переменных $g_n(\omega) = g(n\ \delta_k, \omega)$, $n\in z$ был δ -полупериодичным или слабо полупериодичным, для любого $K\in N^*$

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